Real Analysis

Exercises and Solutions

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実解析(測度ルベーグ積分)

演習問題と解答

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Preface

I wrote this book to review Real Analysis for myself. I picked up important points in a textbook of Real Analysis and rewrote it into an exercise book. There are some mistakes and leaps of logic in the original textbook. I modified them based on my understanding.

はじめに

私が、中国の実解析の教科書を用いて勉強していた際に、重要なポイントを整理して、この問題集の形にまとめあげた。 その教科書内の証明には誤りがあったり、論理の飛躍があったため、私の理解に基づいて可能な限り修正をしている。 また、その教科書より優れたor簡潔な証明があれば、そちらを採用した。 さらに元の教科書の定理が強すぎる条件を与えていた場合、定理自体を書き換えたものもある。 ただし私の書いた解答にも大なり小なり誤りが多く含まれていると思うので、気づいた方は教えていただければ幸いである。

本書は基本的な解析や、集合およびユークリッド空間上の位相の復習からはじまる。 次にユークリッド空間上の点集合に対するルベーグ外測度、およびルベーグ可測性を定義する。 その後、ルベーグ可測関数と、ルベーグ積分を定義した後、ルベーグ積分の性質について議論していく。 また最後に微分やL^p空間についても触れる。

本書はすでにルベーグ積分を勉強した人のための演習書のようだが、しっかりと手を動かしながら理解したい初学者向けの入門書として用いることができるのではないかと思う。 「百聞は一見に如かず」という言葉があるように、数学も説明を聞くよりも、実際に自分で問題を解いてみることが習熟への近道であると思う。 そこで問題演習の形式で、定義、定理、例題の解法を一つ一つ理解しながら読み進められるようにした。

序言

在學習實分析時，我將自己所研讀的課本之重點整理出來，並重新寫爲這本題庫。在定理證明中，我盡量修正了原始課本的錯誤之處，以及針對邏輯跳躍之處加以補充說明。 如果想到比原始課本更簡明易懂的證明，我便採用了該方法。 另外，我發現在原始課本中，有些定理給的條件過強，故我也修改了該書中一些定理的前提條件。 如果本書中有錯誤之處，歡迎讀者們指教與分享。

本書從基本的微積分、歐氏空間上的點集合以及拓撲談起，緊接著定義歐氏空間勒貝格外測度以及勒貝格可測性。 再來，我們定義勒貝格可測函數以及勒貝格積分，然後開始探討勒貝格積分的各種性質。 另外，我們也會探討微分和L^p空間。

本書並不是針對曾學過實分析的人所撰寫的題庫，而本書的主要對象是希望能確實理解實分析的初學者。 俗話說：百聞不如一見，同樣地，當學習數學時，與其專心聽老師講解，不如自己拿起筆多寫題目。 有鑑於此，本書採用了習題演練的形式，讀者們可藉此深度理解書中出現的定義、定理、習題解法等等。

Contents

Part I

Exercises

CHAPTER 1

Set Theory and Point Set

§ 1.1

1 **(Definition 1.17, 1.18, 1.19, 1.20, 1.21)** Answer the following questions.

(1) Let $E \subset \mathbb{R}^d$. Define $\operatorname{diam}(E)$.

(2) Explain what is a bounded set.

(3) Let $x_0 \in \mathbb{R}^d$. Let $\delta > 0$. Define an open ball and a closed ball. We denote them $B(x_0, \delta)$ and $C(x_0, \delta)$ respectively.

(4) An open rectangle. A closed rectangle. A half-open rectangle.

(5) Let $\{x_k\}_{k \geqq 1}$ be a sequende of points on $\mathbb{R}^d$. Define $\lim_{k\to\infty} x_k = x$.

2 **(Definition 1.21, 1.22, 1.23, 1.24, 1.25)** Let $E \subset \mathbb{R}^d$. Answer the following questions.

(1) What is an accumulation point or a limit point of E? We denote a set of limit points of E as E'. What is a closure of E?

(2) What is an isolated point of E. Explain that the set of isolated points of E is expressed as $E \setminus E'$.

(3) What is a closed set? What is a closure of E. (We denote it as $\bar{E}$.)

(4) What is an open set? (State the definition of an open set based on the definition of a closed set.)

(5) What is an interior point of E? (We denote a set of interior points of E as $\mathring{E}$)

(6) What is a boundary of E? We denote a boundary of E as ∂E. Define ∂E based on $\overline{E}$ and $\mathring{E}$. Also show that

$$\partial E = A \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid \forall \delta > 0, B(x, \delta) \cap E \neq \emptyset, B(x, \delta) \cap E^c \neq \emptyset\}.$$

3 **(Theorem 1.13)** Suppose that $E \subset \mathbb{R}^d$. Show that $x \in E'$ if and only if

$$\forall \delta > 0, B(x, \delta) \cap E \setminus \{x\} \neq \emptyset.$$

4 **(Theorem 1.14)** Let $E_1, E_2 \subset \mathbb{R}^d$. Show that

$$(E_1 \cup E_2)' = E_1' \cup E_2'.$$

5 **(Theorem 1.15 Bolzano-Weierstrass Theorem on $\mathbb{R}^d$)** Show that any bounded infinite set $E \subset \mathbb{R}^d$ has at least one limit point. (or if $\{x_n\}_{n \geqq 1} \subset \mathbb{R}^d$ is bounded, we can find a subsequence n_k s.t x_{n_k} converges to some $x \in \mathbb{R}^d$.) You may directly use Bolzano-Weierstrass Theorem on $\mathbb{R}^1$.

6 **(Theorem 1.15 Supplement)** Show Bolzano-Weierstrass Theorem on $\mathbb{R}^1$.

7 **(Exercise 1.4.1)** Let $E \subset \mathbb{R}$ be an uncountable set. Show that $E' \neq \emptyset$.

8 **(Exercise 1.4.2)** Let $E \subset \mathbb{R}^d$ and suppose that E' is a countable set. Show that E is also a countable set.

9 **(Exercise 1.4.5)** Let $E \subset \mathbb{R}^2$ and suppose $\forall x_1, x_2 \in E, |x_1 - x_2| > 1$. Show that E is a countable set.

§ 1.2

(I) Closed Set

10 **(Example 2 and 6)** Let $f(x)$ be a function defined on $\mathbb{R}^d$. Show that $f(x) \in C(\mathbb{R}^d)$ if and only if E_1, E_2 are closed for all $t \in \mathbb{R}$ where

$$E_1 \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid f(x) \geqq t\},\ E_2 \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d | f(x) \leqq t\}$$

(How about open sets?)

11 **(Example 3)** Let $B(x_0, r) \subset \mathbb{R}^d$. Show that the closure of $B(x_0, r)$ is a closed ball $C(x_0, r)$.

If $A \subset E$ and $\overline{A} = A \cup A' = E$, then we say that A is dense in E. In the following examples, we prove that a set is dense. It is enough for us to prove that $\forall \epsilon > 0$ and $\forall x \in E$, there exists $a \in A$ s.t $|x - a| < \epsilon$. (Then we can find $\{a_n\}_{n \geqq 1} \subset A$ s.t $a_n \to x$. So $x \in A'$.)

12 **(Example 4)** Let $a \notin \mathbb{Q}, E_a = \{p + aq | p, q \in \mathbb{Z}\}$. Show that $\bar{E}_a = \mathbb{R}$.

13 **(Example 5)** Let $E = \{\cos n\}$. Show that $\bar{E} = [-1, 1]$. Hint. Use the conclusion of Example 4. $\cos(n + 2m\pi) = \cos n$

14 **(Theorem 1.16 Some Properties of a Closed Set)**

(1) If $F_1, F_2 \subset \mathbb{R}^n$ are closed sets. Then $F_1 \cup F_2$ is a closed set.

(2) If $\{F_\alpha | \alpha \in I\}$ is a family of closed sets, then $F = \bigcap_{\alpha \in I} F_\alpha$ is a closed set.

15 **(Theorem 1.17 Cantor' Intersection Theorem)** Let $\{F_k\}_{k \geqq 1}$ be a sequence of nonempty and bounded closed sets on $\mathbb{R}^d$. Suppose $F_1 \supset F_2 \cdots F_k \supset \cdots$. Show that

$$\bigcap_{k=1}^{\infty} F_k \neq \emptyset.$$

16 **(Exercise 1.5.1.4)** Let $E \subset \mathbb{R}^d$. Show that

$$\bar{E} = \bigcap_{F \supset E; F:\text{ closed}} F.$$

17 **(Exercise 1.5.1.5)** Let $F \subset \mathbb{R}$ be a bounded closed set. Let $f(x)$ be a real-valued function defined on F. For each $x_0 \in F'$, we have $\lim_{x \to x_0, x \in F} f(x) = +\infty$. Show that F is a countable set. Hint. Consider the contraposition. Suppose that F is uncountable and derive a contradiction.

18 **(Exercise 1.5.1.6)** Let $f \in C(\mathbb{R})$. Show that $F = \{(x, y) | f(x) \geqq y\}$ is a closed set on $\mathbb{R}^2$.

(II) Open Set

19 **(Theorem 1.18 Some Properties of an Open Set)**

(1) Let $\{G_\alpha\}_{\alpha \in I}$ be a family of open sets. Show that $G = \bigcup_{\alpha \in I} G_\alpha$ is also an open set.

(2) Let $G_1, G_2 \cdots G_m$ be open sets. Show that $\bigcap_{k=1,2\cdots m} G_k$ is an open set.

(3) Let G be a non-empty set no $\mathbb{R}^d$. G is open if and only if $\forall x \in G, \exists \delta_x$ s.t $B(x, \delta) \subset G$.

20 **(Example 7)** Suppose that $f(x)$ is defined on $B(x_0, \delta_0)$. Let

$$\omega_f(x_0) = \lim_{\delta \to 0} \sup_{x_1, x_2 \in B(x_0, \delta)} \{|f(x_1) - f(x_2)|\}.$$

Show that if G is an openset and f is defined on G, then

$$H = \{x \in G \mid \omega_f(x) < t\}$$

is an open set.

21 **(Theorem 1.19)**

(1) Let G be a non-empty open set on $\mathbb{R}$. It can be expressed as a union of disjoint open intervals.

(2) Let G be a non-empty open set on $\mathbb{R}^d$. It can be expressed as a union of disjoint half open rectangles.

22 **(Exercise 1.5.2.1)** Let $E \subset \mathbb{R}^d$. Show that $\mathring{E} = \left(\overline{(E^c)}\right)^c$.

23 **(Exercise 1.5.2.3)**

(1) Show that G is open $\Leftrightarrow G \cap \partial G = \emptyset$.

(2) Also show that F is closed $\Leftrightarrow \partial F \subset F$.

24 **(Exercise 1.5.2.4)** Let $G \subset \mathbb{R}^d$ be a non-empty open set. Let $r_0 > 0$. Show that $A = \bigcup_{x \in G} \overline{B(x, r_0)}$ is an open set.

25 **(Exercise 1.5.2.5)** Let $F \subset \mathbb{R}$ be an infinite closed set. Show that we can find a countable subset $E \subset F$ s.t $\bar{E} = F$.

26 **(Definition 1.26, Lemma 1.20 Lindelof's Covering Lemma)**

(1) Explain open cover and sub cover.

(2) Let $E \subset \mathbb{R}^d$ be an openset. Suppose $\mathscr{A} = \{A_1, A_2 \cdots\}$ is a family of open balls with $B(y, q)$ where $y \in \mathbb{Q}^d, q \in \mathbb{Q}$. (Hence $\mathscr{A}$ is countable.) Let $x \in E$. Show that we may find $A \in \mathscr{A}$ s.t $x \in A \subset E$.

(3) Suppose $E \subset \bigcup_{\alpha \in I} G_\alpha$. We can always find a countable subset of $I' \subset I$ s.t

$$E \subset \bigcup_{\alpha \in I'} G_\alpha.$$

This is called Lindelof's coverling lemma.

27 **(Theorem 1.21 Heine-Borel's Finite Covering Theorem)** State and Prove Heine-Borel's Covering Theorem.

28 **(Example 8)** Let $F \subset \mathbb{R}^d$ be a bounded closed set. And let $G \subset \mathbb{R}^d$ be an open set. Suppose $F \subset G$. Show that $\exists \delta > 0$ such that $F + \{x\} = \{y + x | y \in F\} \subset G$ for all $x \in (-\delta, \delta)$.

29 **(Theorem 1.22)** Let $E \subset \mathbb{R}^d$. Suppose all open cover of E has finite cover. Show that E is a bounded closed set.

30 **(Exercise 1.5.2.9)** Let $F \subset \mathbb{R}$ be a nonempty countable closed set. Show that F contains at least one isolated point.

31 **(Exercise 1.5.2.10)** Let $f_n(x)$ be a nonnegative decreasing sequence of continuous functions. Suppose there is a closed and bounded set $F \subset \mathbb{R}$ on which $f_n(x) \to 0 (n \to \infty)$. Show that $f_n(x)$ uniformly converges on F.

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We have already considered continuity of a function defined on whole $\mathbb{R}^d$. Now we consider continuity of a function defined on a subset of $\mathbb{R}^d$.

32 **(Definition 1.27)** Let $f(x)$ be a real-valued function defined on $E \subset \mathbb{R}^d$. Let $x_0 \in E$. What does it mean if we say that $f(x)$ is continuous at x_0, and $f(x)$ is continuous on E.

33 **(Example 9)** Let $F \subset \mathbb{R}$ be a bounded and closed set. Let $f(x) : F \to F$. Suppose $|f(x) - f(y)| < |x - y|, x, y \in F$. Show that there exists a fixed point, that is $x_0 \in F$ s.t $f(x_0) = x_0$.

34 **(Exercise 1.5.2.11)** Let $F \subset \mathbb{R}$ be a closed set and $f(x) \in C(F)$. Show that

$$\{x \in F \mid f(x) = 0\}$$

is a closed set.

35 **(Exercise 1.5.2.12)** Let $f(x) : \mathbb{R} \to \mathbb{R}$ and $E_n \subset \mathbb{R}$, $E_n \in \mathscr{O}^1$ (open set), $f(x) \in C(E_n)$. Show that $f(x) \in C\left(\bigcup_{n=1}^{\infty} E_n\right)$.

36 **(Exercise 1.5.2.13)** Let $E \subset \mathbb{R}$.

(1) Suppose $\forall f(x) \in C(E)$ is bounded. Show that E is bounded and closed.

(2) Suppose that every $f(x) \in C(E)$ takes a maximum value on E. Show that E is bounded and closed.

37 **(Exercise 1.5.3.14)** Let $E \subset \mathbb{R}^d$ and let $f : E \to \mathbb{R}$. Suppose $\forall K \subset E$ (K is bounded and closed), we have $f(x) \in C(K)$. Show that $f(x) \in C(E)$.

(III) Borel Sets

38 **(Definition 1.28)** Explain F_σ-sets and G_δ sets.

39 **(Example 11)** Suppose $f(x)$ is a real-valued function defined on an openset $G \subset \mathbb{R}^d$. Show that continuous points of $f(x)$ is a G_δ set.

40 **(Example 12)** Let $\{f_k(x)\} \subset C(\mathbb{R}^d)$ and suppose that $\lim_{k\to\infty} f_k(x) = f(x), \forall x \in \mathbb{R}^d$. Express the set of continuous points of f and show that it is a G_δ set.

41 **(Definition 1.29 1.30, 1.31)**

(1) What is a σ-algebra?

(2) What is a σ-algebra generated from Σ?

(3) What is a Borel set?

42 **(Exercise 1)** Let $\{f_n(x)\}_{n\geqq 1} \subset C([a, b])$ (a sequence of continuous functions on

$[a,b]$) and suppose that $\lim_{n\to\infty} f_n = f (\forall x \in [a,b])$. Show that $\forall t \in \mathbb{R}$,

$$\{x \in [a,b] \mid f(x) < t\}$$

is a F_σ set (a countable union of closed sets).

43 **(Exercise 2)** Let $\{f_n(x)\}_{n \geqq 1} \subset C(F)$ and let $F \subset \mathbb{R}$ be a closed set. Show that

$$\{x \in F \mid f_n(x) \text{ converges}\}$$

is a $F_{\sigma,\delta}$ set.

44 **(Exercise 3)** Let $f(x) : \mathbb{R}^1 \mapsto \mathbb{R}^1$. Show that

$$\left\{x \in \mathbb{R}^1 \mid \lim_{y\to x} f(y) \text{ exists}\right\}.$$

is a G_δ set (a countable intersection of open sets).

45 **(Theorem 1.23 Baire)** Let $E \subset \mathbb{R}^d$ be a F_σ set. Hence $E = \bigcup_{k=1}^{\infty} F_k$. Show that if every $\{F_k\}_{k \geqq 1}$ has no interior point, then E also has no interior point.

46 **(Example 13)** Show that $\mathbb{Q}$ is not a G_δ set.

47 **(Definition 1.32)**

(1) What is a dense set ?

(2) What is a nowhere dense?

(3) What is a meagre set? (This is also called a set of first category. And what is a set of second category?)

48 **(Example 14)** Let $\{G_k\}$ be a sequence of open and dense sets on $\mathbb{R}^d$. Show that $\bigcap_{k=1}^{\infty} G_k$ is dense on $\mathbb{R}^d$.

49 **(Example 15)** Let $f_k \in C(\mathbb{R}^d)$. Suppose that $\lim_{k\to\infty} f_k(x) = f(x) (\forall x \in \mathbb{R}^d)$. Show that the set of discontinuous points of $f(x)$ is a meagre set.

(IV) Cantor Set

50 **(Cantor Set: Definition and Properties)** Let C be a Cantor-Set.

(1) Show that C is a non-empty bounded and closed set.

(2) Show that $C = C'$. (This is called a perfect set.)

(3) Show that C has no interior point.

*

Let us consider the Cantor function $\Phi(x)$. The Cantor function is defined on $[0,1]$ and it has an interesting property. In the next chapter, we will introduce a concept of measure. A Cantor set C defined on $[0,1]$ has a zero measure. The Cantor function is constant on $[0,1] \setminus C$, however the Cantor function is continuous on $[0,1]$.

51 **(Example 17 Cantor function)**

(1) Define the Cantor function (or Cantor-Lebesgue function) $\Phi(x)$.

(2) Show that the Cantor function is continuous.

52 **(Example 18)** Let $E \subset \mathbb{R}$. Show that E is a perfect set if and only if

$$E = \left(\bigcup_{k=1}^{\infty} (a_k, b_k) \right)^c,$$

where $(a_i, b_i), (a_j, b_j)$ $(i \neq j)$ has no common edge point.

53 **(Example 19)** Let $E \subset \mathbb{R}^2$ be a non-empty perfect set. Show that E is an uncountable set.

54 **(Exercise 1)** Let $E \subset \mathbb{R}$ be a non-empty perfect set. Show that $\forall x \in E, \exists y \in E$ s.t $x - y \notin \mathbb{Q}$.

55 **(Exercise 4)** Construct a set of isolated points E such that E' is a perfect set.

§ 1.3

56 **(Definition 1.33 and Theorem 1.24)**

(1) Define $\text{dist}(E_1, E_2)$.

(2) Suppose $F \subset \mathbb{R}^n$ is a non-empty closed set and $x_0 \in \mathbb{R}^n$. Show that $\exists y_0 \in F$ such that $|x_0 - y_0| = \text{dist}(x_0, F)$.

57 **(Theorem 1.25)** Suppose that $E \subset \mathbb{R}^d$ is a non-empty. Let $d(x, E) : \mathbb{R}^n \to [0, \infty)$ be a function of x. Show that $d(x, E)$ is uniformly continuous on $\mathbb{R}^n$.

58 **(Corollary 1.26)** Let $F_1, F_2 \subset \mathbb{R}^d$ be non-empty closed sets and at least one of them is bounded. Show that there exists $x_1 \in F_1, x_2 \in F_2$ s.t

$$|x_1 - x_2| = \text{dist}(F_1, F_2).$$

59 **(Example 2)** Let $F_1, F_2 \subset \mathbb{R}^d$ be disjoint non-empty closed sets. Show that there exists a continuous function $f(x)$ defined on $\mathbb{R}^d$ with

- $0 \leqq f(x) \leqq 1 (x \in \mathbb{R}^d)$
- $F_1 = \{x \in \mathbb{R}^d \mid f(x) = 1\}$ and $F_2 = \{x \in \mathbb{R}^d \mid f(x) = 0\}$.

60 **(Theorem 1.27 Continuous Topology Theorem)** Suppose that $F \subset \mathbb{R}^d$ be a closed set and $f(x)$ is a continuous function defined on F and $|f(x)| \leqq M(x \in F)$. Show that there exists a function $g(x)$ defined on $\mathbb{R}^d$ with

- $g(x) \in C(\mathbb{R}^d)$, ($g(x)$ is continuous on $\mathbb{R}^d$)
- $|g(x)| \leqq M$, ($\forall\ x \in \mathbb{R}^d$)
- $g(x) = f(x)$, ($\forall\ x \in F$) .

61 **(Extension of Theorem 1.27)** Suppose that $F \subset \mathbb{R}^d$ be a closed set and $f(x)$ is a continuous function defined on F. ($f(x)$ is not necessarily bouded on F.) Show that there exists a continuous function $g(x) \in C(\mathbb{R}^d)$ with $f(x) = g(x)$ for all $x \in F$.

62 **(Exercise 1)** Let $E \subset \mathbb{R}^d$ be a nonempty set. Suppose $\forall x \notin E, \exists y \in E$ s.t $|x-y| = \text{dist}(x, E)$. Show that E is a closed set.

63 **(Exercise 2)** Let $G \subset \mathbb{R}^d$ be an open set. Let F be a bounded closed set with $F \subset G$. Show that there exists $r > 0$ such that

$$\{x \mid \text{dist}(x, F) < r\} \subset G.$$

§Exercise

64 **(Exercise 8)** Let $f(x) : \mathbb{R} \to \mathbb{R}$. Suppose $\forall x_0 \in \mathbb{R}$, $\exists \delta > 0$ such that $x \in B(x_0, \delta) \Rightarrow f(x) \geqq f(x_0)$. Show that

$$E \stackrel{\text{def}}{=} \{y = f(x) \mid x \in \mathbb{R}\}$$

is a countable set.

65 **(Exercise 9)** Let $E \subset \mathbb{R}^3$. Suppose $\forall x, y \in E$, $|x-y| \in \mathbb{Q}$. Show that E is countable.

66 **(Exercise 11)** Let $\{f_\alpha(x)\}_{\alpha \in I}$ be a family of real-valued functions defined on $[a, b]$. Suppose $\exists M > 0$ s.t $|f_\alpha(x)| \leqq M(\forall x \in [a, b], \forall \alpha \in I)$. Show that $\forall E \subset [a, b]$ (E : countable), there exists a sequence of functions $\{f_{\alpha_n}(x)\}$ such that $\lim_{n\to\infty} f_{\alpha_n}(x)$ exists for all $x \in E$.

67 **(Exercise 13)** Let $f(x)$ be a monotone increasing function defined on $\mathbb{R}$. Show that E is a closed set.

$$E = \{x : \forall \epsilon > 0, f(x+\epsilon) - f(x-\epsilon) > 0\}$$

68 **(Exercise 14-1)** Let $F \subset \mathbb{R}^d$ be bounded and closed. Let $E \subset F$ be an infinite subset of F. Show that $E' \cap F \neq \emptyset$.

69 **(Exercise 14-2)** Let $F \subset \mathbb{R}^d$. Suppose $\forall E \subset F$ (E : infinite), $E' \cap F \neq \emptyset$. Show that F is bounded and closed.

70 **(Exercise 15)** Let $F \subset \mathbb{R}^d$ be a closed set and let $r > 0$. Show that E is a closed set.

$$E = \{t \in \mathbb{R}^d | \exists x \in F \text{ s.t } |t - x| = r\}.$$

71 **(Exercise 17)** Let $E \subset \mathbb{R}^2$. Let $E_y = \{x \in \mathbb{R} | (x, y) \in E\}$. (This is called a projection set.) Show that $E \subset \mathbb{R}^2$ is closed $\Rightarrow$ E_y is also closed.

72 **(Exercise 18)** Let $f \in C(\mathbb{R})$ and let $\{F_k\}_{k \geqq 1}$ be a decreasing sequence of compact sets. Show that

$$f\left(\bigcap_{k=1}^{\infty} F_k\right) = \bigcap_{k=1}^{\infty} f(F_k).$$

73 **(Exercise 19)** Suppose that $f(x)$ has intermediate value property on $\mathbb{R}$. If $f(x_1) < f(x_2)$ then there exists $c \in (f(x_1), f(x_2))$ and $x_0 \in (x_1, x_2)$ or (x_2, x_1) s.t $c = f(x_0)$. We also suppose $\forall r \in \mathbb{Q}$, $\{x \in \mathbb{R} || f(x) = r\}$ is a closed set. Show that $f(x) \in C(\mathbb{R})$.

74 **(Exercise 20)** Let E_1, E_2 be non-empty sets on $\mathbb{R}$. Suppose $E_2' \neq \emptyset$. Show that

$$\overline{E_1} + E_2' \subset (E_1 + E_2)'.$$

(Notice: $A + B = \{x + y | x \in A, y \in B\}$)

75 **(Exercise 21)** Let $E \in \mathbb{R}^n$. Suppose $E, E^c \neq \emptyset$. Show that $\partial E \neq \emptyset$.

76 **(Exercise 22)** Let $G_1, G_2 \subset \mathbb{R}^2$ be disjoint open sets. Show that $G_1 \cap \overline{G_2} = \emptyset$.

77 **(Exercise 23)** Let $G \subset \mathbb{R}^d$. For any $E \subset \mathbb{R}^d$, we have $G \cap \bar{E} \subset \overline{G \cap E}$. Show that G is an open set.

78 **(Exercise 25)** Let $f : \mathbb{R} \to \mathbb{R}$. Let $G_1 = \{(x, y) \in \mathbb{R}^2 | y < f(x)\}$ and $G_2 = \{(x, y) \in \mathbb{R}^2 | y > f(x)\}$. Show that

$$f(x) \in C(\mathbb{R}) \ \Leftrightarrow \ G_1, G_2 \in \mathscr{O}^1,$$

where $\mathscr{O}^1$ is a collection of all open sets on $\mathbb{R}^1$.

79 **(Exercise 27)** Let $\{F_\alpha\}_{\alpha \in I}$ be a family of bounded closed sets on $\mathbb{R}^d$. For any finite number of closed sets $\{F_{\alpha_1}, F_{\alpha_2}, \cdots, F_{\alpha_m}\} \subset \{F_\alpha\}_{\alpha \in I}$, $\bigcap_{k=1}^m F_{\alpha_k} \neq \emptyset$. Show that

$$\bigcap_{\alpha \in I} F_\alpha \neq \emptyset.$$

80 **(Exercise 28)** Let $\{F_\alpha\}_{\alpha \in I}$ be a family of bounded closed sets on $\mathbb{R}^d$, and let G be an open set on $\mathbb{R}^d$ with $\bigcap_{\alpha \in I} F_\alpha \subset G$. Show that we can find a finite number of closed sets $\{F_{\alpha_1}, \cdots, F_{\alpha_m}\}$ s.t

$$\bigcap_{i=1}^{m} F_{\alpha_i} \subset G.$$

81 **(Exercise 29)** Let $K \subset \mathbb{R}^d$ be a bounded and closed set. Let $\{G_k\}_{k \geqq 1}$ be an open cover of K. Show that $\exists \epsilon_0 > 0$ s.t $\forall x_0 \in K$, $\exists k_0 \in \mathbb{N}$ s.t $B(x_0, \epsilon_0) \subset G_{k_0}$.

82 **(Exercise 30)** Let $f(x)$ be differentiable on $\mathbb{R}$. Moreover suppose that $\forall t \in \mathbb{R}$, $\{x \in \mathbb{R} | f'(x) = t\}$ is closed. Show that $f'(x) \in C(\mathbb{R})$.

83 **(Exercise 31)** Let $f(x) \in C(\mathbb{R})$ be a continuous function on $\mathbb{R}$ with

$$|f(x) - f(y)| \geqq a|x - y|, \ (\forall x, y \in \mathbb{R}),$$

for some $a > 0$. Show that $R(f) \overset{\text{def}}{=} \{f(x) \mid x \in \mathbb{R}\} = \mathbb{R}$. Hint. Show that $R(f)$ is open and closed.

84 **(Exercise 32)** Let $E \subset \mathbb{R}$ be a countable dense set. Show that E is not a G_δ set.

85 **(Exercise 34)** Let $f(x) : \mathbb{R} \to \mathbb{R}$. Suppose that $f(x)$ is continuous at $x \in \mathbb{Q}$ and discontinuous at $x \in \mathbb{R} \setminus \mathbb{Q}$. Show that there does not exist such a function.

86 **(Exercise 37)** Show that every closed set on $\mathbb{R}^d$ is a G_δ set, and also show that every open set on $\mathbb{R}^d$ is a F_σ set.

87 **(Exercise 38)** Let $f(x) : [0, 1] \to \mathbb{R}^1$. Suppose $G_f = \{(x, f(x)) \mid x \in [0, 1]\}$ is a bounded and closed set on $\mathbb{R}^2$. Show that $f(x) \in C([0, 1])$ (continuous on $[0, 1]$).

88 **(Exercise 39)** Let $F \subset \mathbb{R}$. Suppose that $\forall f(x) \in C(F)$, there exists a continuous extension to $\mathbb{R}$. (i.e There exists $g(x) \in C(\mathbb{R}^d)$ s.t $f(x) = g(x)$ for $x \in F$.) Show that F is a closed set.

CHAPTER 2

Lebesgue Measure

§ 2.1 Lebesgue outer measure

When I is an open rectangle on $\mathbb{R}^d$, that is $I \overset{\text{def}}{=} \prod_{i=1}^d (a_i, b_i) = \{(x_1, x_2, \cdots x_d) \mid x_i \in (a_i, b_i)\}$, we define $|I| \overset{\text{def}}{=} \prod_{i=1}^d (b_i - a_i)$.

1 **(Definition 2.1)** Let $E \subset \mathbb{R}^d$. If $\{I_k\}_{k \geqq 1}$ be a collection of a countable number of (or a finite number of) open rectangles. Define Lebesgue outer measure $m^*(E)$.

2 **(Example 1)** Let $x_0 \in \mathbb{R}^d$. Show that

$$m^*(\{x_0\}) = 0.$$

3 **(Example 2)** Let $I = \prod_{i=1}^d (a_i, b_i)$ be an open rectangle on $\mathbb{R}^d$. Then $\bar{I} = \prod_{i=1}^d [a_i, b_i]$ is a closed rectangle. In this question, we may use the fact that if $I \subset \bigcup_{i=1}^k I_i$ then $|I| \leqq \sum_{i=1}^k |I_i|$, where $\{I_i\}_{i=1}^k \cup \{I\}$ are open rectangles and k is finite.

(1) Show that

$$m^*(\bar{I}) = |I|\,.$$

(2) Show that

$$m^*(I) = |I|\,.$$

4 **(Theorem 2.1 Properties of Lebesgue outer measure on $\mathbb{R}^d$)**

(1) Show that m^* is nonnegative, that is $m^*(E) \geqq 0$ and $m^*(\emptyset) = 0$.

(2) Show that m^* is monotone, that is $A \subset B \Rightarrow m^*(A) \leqq m^*(B)$.

(3) Show that m^* has subadditivity, that is $m^*(\bigcup_{k \geqq 1} A_k) \leqq \sum_{k \geqq 1} m^*(A_k)$.

5 **(Corollary 2.2)** Show that $E \subset \mathbb{R}^d$ and E is a countable set $\Rightarrow m^*(E) = 0$.

6 **(Lemma 2.3)** Let $E \subset \mathbb{R}^d$ and let $\delta > 0$. We define $m_\delta^*(E)$ in the following way.

$$m_\delta^*(E) \stackrel{\text{def}}{=} \inf_{\{I_n\}_{n \geqq 1}} \left\{ \sum_{k=1}^{\infty} |I_k| \mid E \subset \bigcup_{k=1}^{\infty} I_k, \text{ edge length of } I_k < \delta \right\}$$

In the definition above, we take infimum with respect to $\{I_k\}_{k \geqq 1}$ where $\{I_k\}_{k \geqq 1}$ is a collection of a countable number of open rectangles covering E whose edge length is less than δ. Show that

$$m_\delta^*(E) = m^*(E).$$

This means that in the definition of outer measure, even if we add a constraint about the edge length of each open rectangle which covers E, the value of outer measure does not change. We use this fact to prove the following theorem.

7 **(Theorem 2.4)**

(1) Let E_1, E_2 be point sets on $\mathbb{R}^d$ and suppose that $\operatorname{dist}(E_1, E_2) > 0$. Show that

$$m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2).$$

(2) Let $\{E_n\}_{n \geqq 1}$ be point sets on $\mathbb{R}^d$ and suppose that $\operatorname{dist}(E_i, E_j) > 0$ for all $i, j \in \mathbb{N}$ $(i \neq j)$. Show that

$$m^* \left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} m^*(E_n).$$

8 **(Theorem 2.5 (a) Translation Invariance)** Let $E \subset \mathbb{R}^d$ and $x_0 \in \mathbb{R}^n$. We define $E_{+x_0} = \{x + x_0 : x \in E\}$. Show that

$$m^*(E_{x_0}) = m^*(E).$$

Hint. Obviously $|I| = |I_{+x}|$.

9 **(Theorem 2.5 (b) Scaling)** Let $E \subset \mathbb{R}^d$ and $\lambda \in \mathbb{R}^d$. We define $\lambda E = \{\lambda x | x \in E\}$. Show that

$$m^*(\lambda E) = |\lambda|^d m^*(E).$$

10 **(Generalized definition of an outer measure)** Let X be a nonempty set and let $\mu^* : 2^X \to [0, \infty]$. Explain μ^* is an outer measure on X.

11 **(Exercise 1)** Let $A \subset \mathbb{R}^d$ and suppose that $m^*(A) = 0$. Let $B \subset \mathbb{R}^d$ be an arbitrary point set. Show that

$$m^*(A \cup B) = m^*(B) = m^*(B \setminus A).$$

12 **(Exercise 2)** Let $A, B \subset \mathbb{R}^d$ and suppose that $m^*(A), m^*(B) < \infty$. Show that

$$|m^*(A) - m^*(B)| \leqq m^*(A \Delta B).$$

13 **(Exercise 3)** Let $E \subset \mathbb{R}^d$. Suppose that $\forall x \in E$, $\exists \delta_x$ s.t $m^*(E \cap B(x, \delta_x)) = 0$. Show that $m^*(E) = 0$.

14 **(Exercise 4)** Let $E \subset [a,b], 0 < c < m^*(E)$. Show that there exists a subset $A \subset E$ s.t $m^*(A) = c$.

15 **(Exercise 5)** Let $C \subset [0,1]$ be a Cantor set. Show that $m^*(C) = 0$.

§ 2.2 Lebesgue measurable sets and Lebesgue measure

We have already defined Lebesgue outer measure of $E \subset \mathbb{R}^d$, $m^*(E)$. In this section, we define Lebesgue measurability based on Lebesgue outer measure $m^*(\cdot)$. If $E \subset \mathbb{R}^d$ is Lebesgue measurable (or simply measurable), its outer measure is often denoted as $m(E) \stackrel{\text{def}}{=} m^*(E)$. (Basically m, m^* have the same meaning. When E is measurable we just prefer to using $m(E)$ than $m^*(E)$.)

16 **(Definition 2.2)** Let $E \subset \mathbb{R}^d$. What does it mean if we say that E is Lebesgue measurable. (or simply measurable.) We denote the family of all Lebesgue measurable sets by $\mathscr{M}$. (i.e $\mathscr{M} \stackrel{\text{def}}{=} \{E \subset \mathbb{R}^d \mid E \text{ is Lebesgue measurable.}\}$.) When we need to emphasize it is on $\mathbb{R}^d$, we sometimes denote it by $\mathscr{M}_d$, $\mathscr{M}^d$ and so on.

17 **(Example 1)** Show that a measure zero set is Lebesgue measurable. (i.e if $m^*(N) = 0$, then $N \in \mathscr{M}$.) This is one of the most important properties of Lebesgue measure.

18 **(Theorem 2.6 Properties of Measurable Sets)** Let $\mathscr{M}$ be a family of Lebesgue measurable sets. Show that following properies.

(1) $\emptyset \in \mathscr{M}$.

(2) $E \in \mathscr{M} \Rightarrow E^c \in \mathscr{M}$.

(3) $E_1, E_2 \in \mathscr{M} \Rightarrow E_1 \cup E_2, E_1 \cap E_2, E_1 \setminus E_2 \in \mathscr{M}$.

(4) $\{E_n\}_{n \geqq 1} \subset \mathscr{M} \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \mathscr{M}$, Moreover, if they are disjoint sets we have $m(\sum_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$. Notice. When E_n are disjoint, we sometimes denote $\bigcup_{n=1}^{\infty} E_n$ as $\sum_{n=1}^{\infty} E_n$.

19 **(Theorem 2.7: continuity of measure)** Let $\{E_k\}_{k \geqq 1}$ is an increasing sequence of Lebesgue measurable sets. Show that

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k \to \infty} m(E_k).$$

20 **(Corollary 2.8: continuity of measure)** Let $\{E_k\}_{k \geqq 1}$ is a decreasing sequence of Lebesgue measurable sets with $m(E_1) < \infty$. Show that

$$m\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k \to \infty} m(E_k).$$

21 **(Example 2: Borel-Cantell's Lemma (I))** Let $\{E_k\}_{k\geqq 1}$ be a sequence of Lebesgue measurable sets. Suppose that $\sum_{k=1}^{\infty} m(E_k) < \infty$. Show that

$$m\left(\limsup_{k\to\infty} E_k\right) = 0.$$

22 **(Corollary 2.9: Fatou's lemma - measure version)** Let $\{E_k\}_{k=1}^{\infty} \subset \mathscr{M}$.

(1) Show that

$$m\left(\liminf_{k\to\infty} E_k\right) \leqq \liminf_{k\to\infty} m\left(E_k\right).$$

(2) Suppose that $m\left(\bigcup_{k=1}^{\infty} E_k\right) < \infty$. Show that

$$\limsup_{k\to\infty} m\left(E_k\right) \leqq m\left(\limsup_{k\to\infty} E_k\right)$$

23 **(Exercise 1)** Let $A \in \mathscr{M}, B \subset \mathbb{R}^d$. B is not necessarily Lebesgue measurable. Show that

$$m^*(A \cup B) + m^*(A \cap B) = m^*(A) + m^*(B).$$

24 **(Exercise 2)** Let $\{A_n\}_{n\geqq 1} \subset \mathscr{M}, B_n \subset A_n$ and suppose that A_n are disjoint. Show that

$$m^*\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} m^*(B_n).$$

25 **(Exercise 3)** Let E_1, E_2 be point sets and let $E_1 \in \mathscr{M}$. Suppose that $m(E_1 \triangle E_2) = 0$. Show that $E_2 \in \mathscr{M}$ and $m(E_1) = m(E_2)$.

26 **(Exercise 4)** Let $\{f_n\}_{n\geqq 1}$ be a sequence of functions defined on $\mathbb{R}^1$ and let $\{\lambda_n\}$ be a sequence of positive numbers. Let $E_n \overset{\text{def}}{=} \{x \in \mathbb{R} \mid |f_n(x)| > \lambda_n\}$. Suppose that $\sum_{n=1}^{\infty} m^*(E_n) < \infty$. Show that there exists a measure zero set Z s.t

$$\limsup_{n\to\infty} \left\{\frac{|f_n(x)|}{\lambda_n}\right\} \leqq 1 \ \forall x \in \mathbb{R} \setminus Z$$

27 **(Exercise 5)** Let $T : \mathbb{R}^d \mapsto \mathbb{R}^d$ be a one to one and onto transformation. Suppose that $m^*(B) = m^*(T(B))$ for all $B \subset \mathbb{R}^d$. Show that

$$T(E) \in \mathscr{M}, \forall E \in \mathscr{M}.$$

28 **(Exercise 6)** Let $X = \{E_\alpha\} \subset \mathscr{M}$ $(E_\alpha \subset \mathbb{R})$ and suppose that $\{E_\alpha\}$ are disjoint and none of them is a measure zero set. Show that X is countable.

29 **(Exercise 7)** Let $\{E_k\}_{k\geqq 1} \subset \mathscr{M}$ and $E_k \subset \mathbb{R}$. Suppose that $E_k \subset [a, b]$ for $k \geqq k_0$ and $\lim_{k\to\infty} E_k = E$. Show that

$$m(E) = \lim_{k\to\infty} m(E_k).$$

30 **(Exercise 8)** Let $E_n \subset [0,1], E_n \in \mathscr{M}, m(E_n) = \epsilon_n$ and suppose that

$$\sum_{n=1}^{\infty} \chi_{E_n}(x) < \infty,\ \forall x \in [0,1] \setminus N,\ m(N) = 0.$$

Show that $\epsilon_n \to 0$.

§ 2.3 Lebesgue measurable sets vs Borel sets

31 **(Lemma 2.10: Caratheodory's Lemma)** Let $G \subset \mathbb{R}^d$, (but $G \neq \mathbb{R}^d$) be an open set and let $E \subset G$. Let $E_k = \{x \in E : \mathrm{dist}(x, G^c) \geqq 1/k\}$. Show that $\lim_{k\to\infty} m^*(E_k) = m^*(E)$.

32 **(Theorem 2.11)** Let F be a nonempty closed set. Show that $F \in \mathscr{M}$.

33 **(Corollary 2.12)** Show that Borel sets are Lebesgue measurable.

34 **(Theorem 2.13)** Let $E \in \mathscr{M}$ and let $\epsilon > 0$ be an arbitrary positive number. Show the following statements.

(1) $\exists G \supset E$ (G: open) s.t $m(G \setminus E) < \epsilon$.

(2) $\exists F \subset F$ (F: closed) s.t $m(E \setminus F) < \epsilon$.

35 **(Converse of Theorem 2.13)** Let $\mathscr{O}^d$ be a collection of all open sets on $\mathbb{R}^d$. Suppose $E \subset \mathbb{R}^d$ satisfies the following condition.

$$\forall \epsilon > 0, \exists G \in \mathscr{O}^d; E \subset G \text{ s.t } m^*(G \setminus E) < \epsilon.$$

Show that $E \in \mathscr{M}$. From these results, we find out that the condition above holds if and only if $E \in \mathscr{M}$. In some textbooks, Lebesgue measurability is defined by the condition above.

36 **(Theorem 2.14)** Let $E \in \mathscr{M}$. Show the following statements.

(1) $\exists H, Z_1$ s.t $E = H \setminus Z_1$ where H: G_δ set and $m(Z_1) = 0$.

(2) $\exists K, Z_2$ s.t. $E = H \cup Z_2$ where $K : F_\sigma$ set and $m(Z_2) = 0$.

37 **(Theorem 2.15: Regularity of Outer Measure)** Let $E \subset \mathbb{R}^d$. Show that there exists a G_δ set H s.t $H \supset E$ and $m(H) = m^*(E)$.

38 **(Corollary 2.16 and 2.17)** Let $\{E_k\}_{k\geqq 1}^{\infty}$ be a sequence of point sets on $\mathbb{R}^d$.

(1) Show that

$$m^*\left(\liminf_{k\to\infty} E_k\right) \leqq \liminf_{k\to\infty} m^*(E_k).$$

(2) Suppose that $\{E_k\}_{k\geqq 1}^{\infty}$ is an increasing sequence. Show that

$$m^*\left(\lim_{k\to\infty} E_k\right) = \lim_{k\to\infty} m^*(E_k).$$

39 **(Theorem 2.18 (a) measurability is translation invariant)** Suppose that $E \in \mathscr{M}$ and $x_0 \in \mathbb{R}^d$. Show that $E_{x_0} = \{x + x_0 : x \in E\} \in \mathscr{M}$ and $m(E_{x_0}) = m(E)$. We have already proven that $m^*(E) = m^*(E_{+x_0})$. This theorem states that Lebesgue measurability is preserved after translation.s

40 **(Theorem 2.18 (b) measurability is scale invariant)** Let $E \subset \mathbb{R}$ and let $\lambda \neq 0$. Show that if $E \in \mathscr{M}$ then $\lambda E \in \mathscr{M}$ where $\lambda E \overset{\text{def}}{=} \{\lambda x \mid x \in E\}$.

41 **(Exercise 1)** Let $E \subset \mathbb{R}^d, m^*(E) < \infty$. Suppose that $m^*(E) = \sup\{m(F)|F \subset E;\ F \text{ is bounded and closed}\}$. Show that $E \in \mathscr{M}$.

42 **(Exercise 2)** Let $E \subset [0,1], E \in \mathscr{M}$.

(1) Suppose that $m(E) = 1$. Show that $\overline{E} = [0,1]$.

(2) Suppose that $m(E) = 0$. Show that $\mathring{E} = \emptyset$.

43 **(Exercise 3)** Let $f(x), g(x)$ be strictly decreasing continuous functions on $[a,b]$. For any $t \in \mathbb{R}$, we have $m(\{x \in [a,b] \mid f(x) > t\}) = m(\{x \in [a,b] \mid g(x) > t\})$. Show that

$$f(x) = g(x) \text{ for all } x \in (a,b).$$

In this question, you may suppose that $\{x \in [a,b] \mid f(x) > t\}$ and $\{x \in [a,b] \mid g(x) > t\}$ are Lebesgue measurable. Actually proof is easy. Since $f(x), g(x)$ are monotone decreasing, $\{x \in [a,b] \mid f(x) > t\}, \{x \in [a,b] \mid g(x) > t\}$ are intervals, thus they are Lebesgue measurable.

44 **(Exercise 4)** Let $E \subset \mathbb{R}$ and suppose that $0 < \alpha < m(E)$. Show that $\exists F \subset E$ (F : bounded and closed) s.t $m(F) = \alpha$.

45 **(Exercise 5)** Let $G \subset \mathbb{R}^1$ be an open set. Does the equality $m(G) = m(\overline{G})$ always hold?

46 **(Exercise 6)** Let $E_1, E_2 \subset \mathbb{R}^d$ and suppose that $E_1 \cup E_2 \in \mathscr{M}$ with $m(E_1 \cup E_2) < \infty$. Show that if

$$m(E_1 \cup E_2) = m^*(E_1) + m^*(E_2),$$

then $E_1, E_2 \in \mathscr{M}$.

47 **(Exercise 7)** Construct a set of second category $E \subset [0,1]$ with measure zero.

48 **(Exercise 8)** Let $A \subset \mathbb{R}$ and for every $x \in A$ there exists infinitely many $(p,q) \in \mathbb{Z} \times \mathbb{N}$ s.t $|x - p/q| \leqq 1/q^3$. Show that $m(A) = 0$.

§ 2.4 Sets of positive measure and Rectangles

49 **(Theorem 2.19)** Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set and suppose that $m(E) > 0$. Let $0 < \lambda < 1$. Show that there exists a rectangle I such that $\lambda|I| < m(I \cap E)$.

50 **(Theorem 2.20 Steinhaus Theorem)** Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set. We suppose that $m(E) > 0$. We define $E - E \stackrel{\text{def}}{=} \{x - y : x, y \in E\}$. Show that there exists $\delta_0 > 0$ s.t. $E - E \supset B(0, \delta_0)$.

51 **(Exercise 1)** Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $m(E) > 0$. Show that there exists $\alpha > 0$ such that $E_{+x} \cap E \neq \emptyset$. $(|x| < a)$ where $E_{+x} \stackrel{\text{def}}{=} \{x + y \mid y \in E\}$

52 **(Exercise 2)** Let $E \subset \mathbb{R}$ be a Lebesgue measurable set, let $a \in \mathbb{R}$ and let $\delta > 0$. Suppose that $\forall x : |x| < \delta$, we have $a + x \in E$ or $a - x \in E$. Show that

$$m(E) \geqq \delta.$$

53 **(Exercise 3)** Let $f(x)$ be a function defined on $\mathbb{R}$. Suppose that $f(x + y) = f(x) + f(y), \forall x, y \in \mathbb{R}$ and $f(x)$ is bounded on $x \in E \subset \mathbb{R}; E \in \mathscr{M}; m(E) > 0$. Show that

$$f(x) = cx, \text{ where } c = f(1).$$

§ 2.5 Lebesgue non-measurable sets

54 **(Example: non Lebesgue measurable set)** Construct a non Lebesgue measurable set.

55 **(Extra Theorem)** Show that if $A \subset \mathbb{R}^d$ with $m^*(A) > 0$ (A is not necessarily measureble) then $\exists W \subset A$ s.t $W \notin \mathscr{M}$.

56 **(Exercise 1)** Discuss if there exists a point set $E \subset [0, 1]$ s.t $\forall x \in \mathbb{R}, \exists y \in E$ s.t $x - y \in \mathbb{Q}$.

57 **(Exercise 2)** Construct a family of disjoint point sets $\{E_k\}_{k \geqq 1}^{\infty}$ s.t

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) < \sum_{k=1}^{\infty} m^*(E_k).$$

58 **(Exercise 3)** Construct an uncountable point set $W \subset [0, 1]$ s.t $W - W$ has no interior point.

59 **(Exercise 4)** Show that $W \notin \mathscr{M}, E \in \mathscr{M} \Rightarrow E \triangle W \notin \mathscr{M}$.

60 **(Exercise 5)** Let E be a point set. Suppose that

$$\sup_{F:\ \text{closed}; F \subset E} \{m(F)\} < \inf_{G:\ \text{open}; E \subset G} \{m(G)\}.$$

Show that E is not Lebesgue measurable.

61 **(Exercise 6)** Let $\{E_\alpha\}_{\alpha \in I} \subset \mathscr{M}$. Prove or disprove $\bigcap_{\alpha \in I} E_\alpha \in \mathscr{M}$. Of course when I is countable, the statements holds. However when I is not countable does the statement still hold?

62 **(Extra Exercise 1)** Let Γ be a family of half open intervals on $\mathbb{R}^1$, that is $\forall I \in \Gamma$, $I = (a, b]$ or $I = [a, b)$. Show that $\bigcup_{I\in\Gamma} I$ is Lebesgue measurable.

63 **(Extra Exercise 2)** Let $f : [a, b] \mapsto \mathbb{R}^1$ be a one-to-one and onto transformation. For all $E \in \mathscr{M}, E \subset [a, b]$, $f(E) \in \mathscr{M}$. Show that

$$m(f(Z)) = 0,\ \forall Z \subset \mathbb{R}^1, m(Z) = 0.$$

§ 2.6 Continuous transformation and Lebesgue measurable sets

64 **(Definition 2.3)** Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a transformation from $\mathbb{R}^d$ to $\mathbb{R}^d$. What does it mean if we say that T is a continuous transformation? State the definition of continuity based on an inverse image of an open set.

65 **(Theorem 2.21)** Show that a transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ is continuous if and only if $\forall x \in \mathbb{R}^d, \forall\epsilon > 0, \exists\delta(x, \epsilon)$ s.t.

$$\forall y \in B(x, \delta), |T(y) - T(x)| < \epsilon.$$

66 **(Example 1)** Let $T : \mathbb{R}^d \to \mathbb{R}^d$. Show that if T is linear, then T is continous.

67 **(Theorem 2.22: Compact Set and Continuous Transformation)** Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous transformation. Suppose that K is a compact set on $\mathbb{R}^d$. Show that $T(K)$ is a compact set on $\mathbb{R}^d$.

68 **(Corollary 2.23, 2.24)** Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous transformation.

(1) Let E be a F_σ set. Show that $T(E)$ is also a F_σ set.

(2) Suppose that $T(Z)$ is a measure zero set for all Z with measure zero. Now let E be a Lebesgue measurable set. Show that $T(E)$ is also a Lebesgue measurable set.

(3) Do all continuous transformations $\mathbb{R}^d \mapsto \mathbb{R}^d$ satisfy $m(T(Z)) = 0,\ \forall Z : m(Z) = 0$?

69 **(Extra Theorem: Lipschitz Continuous)** Let $T : \mathbb{R}^d \mapsto \mathbb{R}^d$.

(1) Explain what is Lipschitz continuity.

(2) Suppose T is Lipschitz continuous. Show that $T(Z) = 0$ for all Z with $m(Z) = 0$. If necessary, you may use the fact that an open ball B on $\mathbb{R}^d$ with radius r has a measure

$$m(B) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)} r^d.$$

This result can be derived by Tonelli's theorem in Chapter 4.

70 **(Theorem 2.25, 2.26)** Suppose that $T : \mathbb{R}^d \to \mathbb{R}^d$ be a nonsingular linear transformation. Let $E \subset \mathbb{R}^d$. Show that $m^*(T(E)) = |\mathrm{det}T|m^*(E)$. Especially, if $E \in \mathscr{M}$, we have $m(T(E)) = |\mathrm{det}T|m(E)$.

71 **(Extra Exercise 1)** Let $f(x)$ be a function defined on $\mathbb{R}$. Suppose we have

$$|f(x) - f(y)| \leqq e^{|x|+|y|} |x - y|, \ \forall x, y \in \mathbb{R}.$$

Show that

$$m(E) = 0 \Rightarrow m(f(E)) = 0.$$

72 **(Extra Exercise 2)** Explain that rotation does not change the value of Lebesgue measure on $\mathbb{R}^2$

§ 2.7 Construction of non-Borel measurable set

73 **(Lemma)** Let $f(x)$ be a real-valued function defined on $E \subset \mathbb{R}^n$. Let Γ be a $\sigma-$algebra that consists of point sets on $\mathbb{R}^n$. Suppose $E \in \Gamma$. Show that

$$\mathscr{A} = \{A \subset \mathbb{R} \mid f^{-1}(A) \in \Gamma\}$$

is a $\sigma-$algebra.

74 **(Corollary)** Let $f(x)$ be a continuous function on $\mathbb{R}$. Let $A \subset \mathbb{R}$ be a Borel set. Show that $f^{-1}(A)$ is also a Borel set.

75 **(Example: non-Borel set)** Construct a non-Borel (or non-Borel measurable) set.

§ 2.8 Exercise

76 **(Exercise 1)** Let $E \subset \mathbb{R}$ and let $q \in (0, 1)$. For any open interal (a, b), we have $\{I_n\}_{n \geqq 1}$ s.t

$$E \cap (a, b) \subset \bigcup_{n=1}^{\infty} I_n, \sum_{n=1}^{\infty} m(I_n) < (b - a)q.$$

Show that $m(E) = 0$.

77 **(Exercise 2)** Let $A_1 \in \mathscr{M}, \mathbb{R}^d \supset A_2 \supset A_1$. Suppose that $m(A_1) = m^*(A_2) < \infty$. Show that $A_2 \in \mathscr{M}$.

78 **(Exercise 4)** Let $F \subset [a, b]$ a closed set and $F \neq [a, b]$. Prove or disprove there exists F s.t $m(F) = b - a$.

79 **(Exercise 5)** Construct a closed set $F \subset \mathbb{R}$ where $\forall x \in F$ is a irrational number and $m(F) > 0$.

80 **(Exercise 7)** Let $\{E_k\}_{k \geqq 1} \subset \mathscr{M}$. Suppose that $m\left(\bigcup_{k=1}^{\infty} E_k\right) < \infty$. Show that

$$m\left(\limsup_{k\to\infty} E_k\right) \geqq \limsup_{k\to\infty} m(E_k).$$

81 **(Exercise 8)** Let $\{E_k\}_{k\geqq 1} \subset \mathscr{M}, E_k \subset [0,1], m(E_k) = 1$. Show that

$$m\left(\bigcap_{k=1}^{\infty} E_k\right) = 1.$$

82 **(Exercise 9)** Let $E_1, E_2 \cdots E_k$ be Lebesgue measurable sets on $[0,1]$. Suppose that $\sum_{i=1}^{k} m(E_i) > k-1$. Show that

$$m\left(\bigcap_{i=1}^{k} E_i\right) > 0.$$

83 **(Exercise 11)** Let $\{B_\alpha\}_{\alpha\in I}$ be a family of open balls on $\mathbb{R}^d$. Let $G = \bigcup_{\alpha\in I} B_\alpha$. Suppose $0 < \lambda < m(G)$. Show there exists finite number of disjoint open balls $\{B_1, B_2 \cdots B_\ell\} \subset \{B_\alpha\}_{\alpha\in I}$ such that

$$\sum_{k=1}^{\ell} m(B_k) > \frac{\lambda}{3^d}.$$

84 **(Exercise 12)** Let $\{B_k\} \subset \mathscr{M}$ be a decreasing sequence of measurable sets. Let $A \subset \mathbb{R}^d : m^*(A) < \infty$. Let $E_k = A \cap B_k$ and let $E = \bigcap_{k=1}^{\infty} E_k$. Show that

$$\lim_{k\to\infty} m^*(E_k) = m^*(E).$$

85 **(Exercise 13)** Let $E \subset \mathbb{R}^d$ $(m^*(E) < \infty)$, $H \supset E, H \in \mathscr{M}$. Suppose that $\forall N \subset H \setminus E$, if $N \in \mathscr{M} \Rightarrow N$ is a measure zero set. Discuss if H is a measurable cover of E. (i.e. $m(H) = m^*(E)$)

86 **(Exercise 14)** Show that $E \in \mathscr{M}$ if and only if $\forall \epsilon > 0$ there exists G_1, G_2 : $G_1 \supset E, G_2 \supset E^c$ s.t $m(G_1 \cap G_2) < \epsilon$.

87 **(Exercise 15)** Let $E \subset [0,1]$ be a Lebesgue measurable set and let $\{x_i\}_{i=1}^{n} \subset [0,1]$. Suppose that $m(E) \geqq \epsilon > 0$ and $n > \frac{2}{\epsilon}$. Show that $\exists y_1, y_2 \in E$ and $\exists i, j \in \{1, 2 \cdots, n\}$ s.t

$$|y_1 - y_2| = |x_i - x_j|.$$

88 **(Exercise 16)** Let $W \subset [0,1]$ be a non measurable set. Show that there exists $\epsilon > 0$ such that for all $E \subset [0,1], E \in \mathscr{M}$ with $m(E) \geqq \epsilon$, we have

$$W \cap E \notin \mathscr{M}.$$

89 **(Extra Exercise 1)** Let $\{r_n\} \overset{\text{def}}{=} \mathbb{Q}$. Let

$$G \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} \left(r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2}\right).$$

Show that

$$m(G \Delta F) > 0,$$

for all closed set $F \subset \mathbb{R}^1$.

90 **(Extra Exercise 2)** Let $\{E_n\}_{n\geqq 1}$ be a sequence of Lebesgue measurable sets and suppose that $\limsup_{n\to\infty} m(E_n) = 1$. Show that for all $\alpha \in (0,1)$ we have a subsequence $\{n_k\}$ s.t

$$m\left(\bigcap_{k=1}^{\infty} E_{n_k}\right) > \alpha.$$

91 **(Extra Exercise 3)** Let $E \subset [0,1]$ be a Lebesgue measurable set with $m(E) > 0$. Show that there exist n disjoint Lebesgue measurable sets $\{E_i\}_{i=1}^{n}$ s.t

$$E = \bigcup_{i=1}^{n} E_i \ ; \ m(E_i) = \frac{1}{n} m(E).$$

92 **(Extra Exercise 4)** Let $E \subset \mathbb{R}^1$ be a Lebesgue measurable set with $m(E) < \infty$. Show that

$$\lim_{x\to\infty} m\left(E_{+x} \cap E\right) = 0.$$

CHAPTER 3

Lebesgue measurable functions

§ 3.1 Lebesgue measurable functions and their properties

1 **(Definition 3.1: Lebesgue measurable function)** Let $f(x) : E \to \overline{\mathbb{R}}$ where $E \subset \mathbb{R}^d, E \in \mathscr{M}$. State the definition for $f(x)$ to be a measurable function (a Lebesgue measurable function) defined on E. (When we discuss a Lebesgue measurable function defined on $E \subset \mathbb{R}^d$, E is implicitly a Lebesgue measurable set.)

2 **(Theorem 3.1)** Let $f(x)$ be a function defined on $E \in \mathscr{M}$. Let $D \subset \mathbb{R}$ be a dense set. Suppose $\forall r \in D$, $\{x \mid f(x) > r\} \in \mathscr{M}$. Show that $\forall t \in \mathbb{R}$, $\{x \mid f(x) > t\} \in \mathscr{M}$.

3 **(Example 1)** Let $f(x)$ be a monotone increasing (or decreasing) function defined on $[a, b]$. Show that $f(x)$ is Lebesgue measurable function defined on $[a, b]$.

4 **(Theorem 3.2)** If $f(x)$ is a Lebesgue measurable function defined on $E \in \mathscr{M}$. Show the following sets are all Lebesgue measurable.

(1) $\{x \in E \mid f(x) \leqq t\}$

(2) $\{x \in E \mid f(x) \geqq t\}$

(3) $\{x \in E \mid f(x) < t\}$

(4) $\{x \in E \mid f(x) = t\}$

(5) $\{x \in E \mid f(x) < \infty\}$

(6) $\{x \in E \mid f(x) = +\infty\}$

(7) $\{x \in E \mid f(x) > -\infty\}$

(8) $\{x \in E \mid f(x) = -\infty\}$

5 **(Theorem 3.3)**

(1) Let $f(x) : E_1 \cup E_2 \to \overline{\mathbb{R}}$ and let $E_1, E_2 \subset \mathbb{R}^d (\in \mathscr{M})$. Suppose that $f(x)$ is measurable on E_1 and E_2. Show that $f(x)$ is measurable on $E_1 \cup E_2$.

(2) Let $f(x)$ be a Lebesgue measurable function on $E \in \mathscr{M}$. Let $A \subset E, A \in \mathscr{M}$. Show that $f(x)$ is a Lebesgue measurable function on A.

6 **(Example 2)** Let $E \subset \mathbb{R}^d; E \in \mathscr{M}$. Show that $\chi_E(x)$ is a Lebesgue measurable function on $\mathbb{R}^d$.

7 **(Theorem 3.4: Properties of Measurable Functions I)** Let $f(x), g(x)$ be real-valued Lebesgue measurable functions on $E \in \mathscr{M}$. (A real-valued function does not take $\infty, -\infty$, so $f(x), g(x) : E \to \mathbb{R}$.) Show that the followings are Lebesgue measurable functions.

(1) $cf(x)(c \in \mathbb{R})$.

(2) $f(x) + g(x)$.

(3) $f(x)g(x)$.

8 **(Corollary 3.5)** Theorem 3.4 holds for $f(x), g(x) : E \to \overline{\mathbb{R}}$. You may assume that $(f(x), g(x)) \neq (+\infty, -\infty), (-\infty, +\infty)$ on E because $f(x)+g(x)$ is not defined in such cases.

9 **(Theorem 3.6, Corollary 3.7: Properties of Measurable Functions II)** Let $\{f_k(x)\}_{k \geqq 1}$ be a sequence of measurable functions. Show that the following items are also measurable functions.

(1) $\sup_{k \geqq 1}\{f_k(x)\}$

(2) $\inf_{k \geqq 1}\{f_k(x)\}$.

(3) $\limsup_{k \to \infty} f_k(x)$.

(4) $\liminf_{k \to \infty} f_k(x)$.

Especially, when $f_k(x) \to f(x)$ exists, $f(x)$ is also measurable.

10 **(Example 3)** Let $f(x)$ be a Lebesgue measurable function defined on $E \in \mathscr{M}$. Show that $f^+(x) \stackrel{\text{def}}{=} \max\{f(x), 0\}$ and $f^-(x) \stackrel{\text{def}}{=} \max\{-f(x), 0\}$ are Lebesgue measurable functions.

11 **(Example 4)** Let $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$. For each $x \in \mathbb{R}$, $y \mapsto f(x, y) : \mathbb{R} \to \mathbb{R}$ is a continuous function. For each $y \in \mathbb{R}$, $x \mapsto f(x, y) : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function on $\mathbb{R}$. Show that $f(x, y)$ is a measurable function on $\mathbb{R}^2$. If necessary, you may suppose that if $A \in \mathscr{M}_1, B \in \mathscr{M}_1$ then $A \times B \stackrel{\text{def}}{=} \{(x, y) \mid x \in A, y \in B\} \in \mathscr{M}_2$. $\mathscr{M}_1, \mathscr{M}_2$ are the collections of all Lebesgue measurable sets on $\mathbb{R}^1$ and $\mathbb{R}^2$ respectively.

12 **(Example 5)** Let $E \subset \mathbb{R}, E \in \mathscr{M}$ and let $f : E \mapsto \mathbb{R} \in C(E)$. Show that $f(x)$ is a Lebesgue measurable function defined on E.

13 **(Exercise 1)** Let $f(x)$ be a function defined on $E \in \mathscr{M}, E \subset \mathbb{R}^d$. Suppose that $f(x)^2$ is measurable on E and $\{x \in E : f(x) > 0\} \in \mathscr{M}$. Show that $f(x)$ is measurable on E.

14 **(Exercise 2)** Let $\mathscr{F}$ be a family of continuous functions defined on $(0,1)$. Show that

$$g(x) \stackrel{\text{def}}{=} \sup\{f \mid f \in \mathscr{F}\},\ h(x) \stackrel{\text{def}}{=} \inf\{f \mid f \in \mathscr{F}\}$$

are measurable functions on defined $(0,1)$.

15 **(Exercise 3)** Let $\{f_k(x)\}_{k \geqq 1}$ be a sequence of measurable functions defined on $E \in \mathscr{M}$. Let $A = \{x \in E : f_k(x) \text{ converges}\}$. Show that $A \in \mathscr{M}$.

16 **(Exercise 4)** Let $f(x)$ be a Lebesgue measurable function defined on E. Let G, F be an open set and a closed set respectively. Show that

$$E_1 \stackrel{\text{def}}{=} \{x \in E \mid f(x) \in G\},\ E_2 \stackrel{\text{def}}{=} \{x \in E \mid f(x) \in F\}$$

are measurable sets.

17 **(Definition 3.2)** Let $E \subset \mathbb{R}^d, E \in \mathscr{M}$. Consider a proposition $P(x)$ related to $x \in E$. What does it mean to say that $P(x)$ is true almost everywhere on E (or $P(x)$ is true for almost every $x \in E$.)

*

In Definition 3.2, let $\{f_k(x)\}_{k \geqq 1} \cup \{f(x)\}$ be a sequence of functions defined on $E \in \mathscr{M}$. (not necessarily measurable functions) Let the proposition $P(x) : f_k(x) \to f(x)$ as $k \to \infty$. If $P(x)$ is true for almost every $x \in E$, then we say that $f_k(x)$ converges to $f(x)$ almost everywhere on E. And we denote it as $f_k(x) \xrightarrow{\text{a.e}} f(x)$ on E or $f_k(x) \to f(x)$ a.e $x \in E$. (a.e is an abbreviation for almost everywhere.)

18 **(Theorem 3.8)** Let $f(x), g(x) : E \to \overline{\mathbb{R}}$ be measurable functions defined on $E \in \mathscr{M}$. Suppose that $f(x) = g(x)$ a.e $x \in E$. Show that $g(x)$ is measurable on E.

19 **(Extra Example)** Let $\{f_k(x)\}_{k \geqq 1}$ be a sequence of Lebesgue measurable functions on E. Let $f(x)$ be a function (not necessarily Lebesgue measurable) defined on E. Suppose that $f_k(x) \xrightarrow{\text{a.e}} f(x)$ on E. Show that $f(x)$ is a measurable function defined on E. In Theorem 3.6, we have shown that if $f_k(x)$ is measurable and $f_k(x) \to f(x)$ (converges at every $x \in E$) then $f(x)$ is also measurable. This example claims that $\to$ can be replaced with $\xrightarrow{\text{a.e}}$.

20 **(Example 6)** Let $0 < m(A) < \infty, A \in \mathscr{M}$ and let $f(x)$ be measurable on $A \subset \mathbb{R}^d$. Suppose that $0 < f < \infty$ a.e $x \in A$. Show that $\forall \delta \in (0, m(A))$, $\exists B \subset A, B \in \mathscr{M}$ and $\exists k_0 \in \mathbb{N}$ such that $m(A \setminus B) < \delta$ and $1/k_0 \leqq f(x) \leqq k_0 (\forall x \in B)$.

21 **(Exercise 6)** Let $f(x) \in C([a,b])$ and let $g(x) : [a,b] \to \mathbb{R}$. Suppose that $g(x) = f(x)$ a.e $x \in [a,b]$ Discuss if $g(x)$ is continuous a.e $x \in [a,b]$.

22 **(Exercise 7)** Let $f(x)$ be a function continuous a.e $x \in \mathbb{R}$. Discuss if there exists $g(x) \in C(\mathbb{R})$ s.t $f = g$ a.e $x \in \mathbb{R}$.

23 **(Definition 3.3: Simple Function)** Explain the following terms.

(1) a simple function

(2) a measurable simple function

(3) a step function

24 **(Theorem 3.9 Approximation Theorem by Simple Functions)** Prove the following statements.

(1) Suppose that $f(x) : E \mapsto [0,\infty]$ is a non-negative Lebesgue measurable function defined on $E \in \mathscr{M}; E \subset \mathbb{R}^d$. Show that there exists an increasing sequence of non-negative Lebesgue-measurable simple functions $\{f_k(x)\}_{k\geqq 1}; 0 \leqq f_k(x) \leqq f(x)$ s.t $\lim_{k\to\infty} f_k(x) = f(x)$ on E.

(2) Suppose that $f(x)$ is a measurable function defined on $E \in \mathscr{M}; E \subset \mathbb{R}^d$. Show that there exists a sequence of Lebesgue-measurable simple functions$\{f_k(x)\}_{k\geqq 1}$: $|f_k(x)| \leqq |f(x)|$ s.t $\lim_{k\to\infty} f_k(x) = f(x)$ on E.

(3) Show that if $f(x)$ is bounded, $f_k(x) \xrightarrow{u} f(x)$ on E.

25 **(Definition 3.4)** Let $f(x)$ be a function defined on E. State the definition of $\mathrm{supp}(f)$.

26 **(Corollary 3.10)** Show that in Theorem 3.9, it is possible for us to suppose that each $f_k(x)$ has a compact support.

§ 3.2 Convergence of Lebesgue measurable functions

27 **(Definition 3.5)** Let $f(x), f_k(x) : E \to \overline{\mathbb{R}}$ and let $E \subset \mathbb{R}^d$. What does it mean to say that $\{f_k(x)\}_{k\geqq 1}$ converges to $f(x)$ almost everywhere on E?

28 **(Lemma 3.11)** Let $\{f(x)\}\cup\{f_k(x)\}_{k\geqq 1}$ be Lebesgue measurable functions finite almost everywhere on $E \in \mathscr{M}$. (i.e $|f_k(x)| < \infty$ a.e $x \in E$ for each $k \in \mathbb{N}$.) Suppose that $m(E) < \infty$ and $f_k(x) \xrightarrow{\text{a.e}} f(x)$ on E. Show that $\forall\epsilon > 0$, we have

$$\lim_{j\to\infty} m\left(\bigcup_{k=j}^{\infty} E_k(\epsilon)\right) = 0$$

where

$$E_k(\epsilon) \stackrel{\text{def}}{=} \{x \in E \mid |f_k(x) - f(x)| \geqq \epsilon\}.$$

*

Before Theorem 3.12, let us introduce a new concept of convergence defined for a sequence of Lebesgue measurable functions. Let $\{f_k(x)\}_{k\geqq 1}$ be a sequence of measurable functions defined on $E \in \mathscr{M}$. (In this definition, both $m(E) < \infty$ and $m(E) = \infty$ are allowed.) If $\forall\delta > 0$, there exists $E_\delta : m(E_\delta) < \delta$, such that $f_k(x) \overset{u}{\to} f(x)$ (i.e converges uniformly) on $E \setminus E_\delta$, then we say that $f_k(x)$ converges to $f(x)$ almost uniformly on E. We denote it as

$$f_k(x) \overset{\text{a.u}}{\longrightarrow} f(x) \text{ on } E$$

29 **(Theorem 3.12 Egorov)** Let $f(x), f_1(x), f_2(x) \cdots$ be Lebesgue measurable functions finite almost everywhere on E. Suppose that $m(E) < \infty$ and $f_k(x) \overset{\text{a.e}}{\longrightarrow} f(x)x \in E$. Show that

$$f_k(x) \overset{\text{a.u}}{\longrightarrow} f(x) \text{ on } E.$$

*

Theorem 3.12 Egorov's theorem states that if $m(E) < \infty$, $f_k(x) \overset{\text{a.e}}{\longrightarrow} f(x)$ on E implies that $f_k(x) \overset{\text{a.u}}{\longrightarrow} f(x)$. However, $f_k(x) \overset{\text{a.u}}{\longrightarrow} f(x)$ on E always implies that $f_k(x) \overset{\text{a.e}}{\longrightarrow} f(x)$ on E without the assumption $m(E) < \infty$. We will prove Egorov's theorem again using another extra theorem, which helps you to clarify the relationship between several different convergence concepts.

30 **(Example 1)** Suppose that $f_n(x) = x^n (0 \leqq x \leqq 1), f(x) = 0 (0 \leqq x < 1), f(1) = 1$. Verify that $f_n(x) \to f(x)$ but not $f_n(x) \overset{u}{\to} f(x)$. ($\overset{u}{\to}$ means uniform convergence.)

31 **(Definition 3.6)** Again we introduce another concept of convergence. Let $\{f_k(x)\}_{k\geqq 1} \cup \{f(x)\}$ be measurable functions defined on $E \in \mathscr{M}$ and all of them are finite almost everywhere on E. What does it mean to say that $f_k(x)$ converges to $f(x)$ in measure on E? We denote it as

$$f_k(x) \overset{m}{\to} f(x) \text{ on } E.$$

*

Let $f(x), g(x)$ be measurable functions defined on $E \in \mathscr{M}$. If $m(\{x \in E \mid f(x) \neq g(x)\}) = 0$, then we say that $f(x)$ and $g(x)$ are equivalent on E.

32 **(Theorem 3.13)** Let $\{f_k(x)\}_{k\geqq 1}$ be a sequence of measurable functions defined on $E \in \mathscr{M}$. Let $f(x), g(x)$ be measurable functions defined on E with $|f(x)|, |g(x)| < \infty$ a.e $x \in E$. Suppose that $f_k(x) \overset{m}{\to} f(x)$ and $f_k(x) \overset{m}{\to} g(x)$ on E. Show that $f(x), g(x)$ are equivalent.

*

The following theorem states that in a finite measure space (i.e if $m(E) < \infty$), $f_k(x) \overset{\text{a.e}}{\longrightarrow} f(x)$ on E implies that $f_k(x) \overset{m}{\to} f(x)$ on E. We can also prove this statement using the extra theorem below, but we first prove the statement using the theorems and the lemma we have introduced.

33 **(Theorem 3.14)** Let $\{f_k(x)\}_{k\geqq 1}$ be a sequence of measurable functions on $E \in \mathscr{M}, m(E) < \infty$ and suppose that $|f_k(x)| < \infty$ a.e $x \in E$. And suppose that $f_k(x) \stackrel{\text{a.e}}{\longrightarrow} f(x) (x \in E)$ where $|f(x)| < \infty$ a.e $x \in E$. Show that $f_k(x) \stackrel{m}{\longrightarrow} f(x)$. (However its converse does not hold.)

*

Until now, we have already introduced three new concepts of convergence related to measurable functions. The following extra theorem will be of great help for you to clarify the relationship between $f_k(x) \stackrel{\text{a.u}}{\longrightarrow} f(x)$, $f_k(x) \stackrel{\text{a.e}}{\longrightarrow} f(x)$ and $f_k(x) \stackrel{m}{\longrightarrow} f(x)$. By using the extra theorem we can easily find out the following facts.

- if $f_k(x) \stackrel{\text{a.u}}{\longrightarrow} f(x)$, then $f_k(x) \stackrel{\text{a.e}}{\longrightarrow} f(x)$ and also $f_k(x) \stackrel{m}{\longrightarrow} f(x)$ without any assmption about $m(E)$. (But $f_k(x) \stackrel{\text{a.e}}{\longrightarrow} f(x)$ does not imply $f_k(x) \stackrel{m}{\longrightarrow} f(x)$ if $m(E) = \infty$.)
- especially, when $m(E) < \infty$, $f_k(x) \stackrel{\text{a.e}}{\longrightarrow} f(x)$ if and only if $f_k(x) \stackrel{\text{a.u}}{\longrightarrow}$. ($\Rightarrow$ is called Egorov's theorem.)

From these facts, if $m(E) < \infty$,

$$f_k(x) \stackrel{\text{a.e}}{\longrightarrow} f(x) \ \Leftrightarrow \ f_k(x) \stackrel{\text{a.u}}{\longrightarrow} f(x) \ \Rightarrow f_k(x) \stackrel{m}{\longrightarrow} f(x).$$

34 **(Extra Theorem: equivalent statements to $\stackrel{\text{a.e}}{\longrightarrow}$ and $\stackrel{\text{a.u}}{\longrightarrow}$)** Let $\{f_k(x)\}_{k\geqq 1}$ be a sequence of Lebesgue measurable functions defined on $E \in \mathscr{M}$ and suppose that $|f_k(x)|, |f(x)| < \infty$ a.e $x \in E$.

(1) $f_k(x) \stackrel{\text{a.e}}{\longrightarrow} f(x)$ on E if and only if

$$m\left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \{x \in E \mid |f_k - f| \geqq \epsilon\}\right) = 0, \forall \epsilon > 0$$

(2) $f_k(x) \stackrel{\text{a.u}}{\longrightarrow} f(x)$ on E if and only if

$$\lim_{m\to\infty} m\left(\bigcup_{k=m}^{\infty} \{x \in E \mid |f_k - f| \geqq \epsilon\}\right) = 0, \forall \epsilon > 0$$

35 **(Theorem 3.15)** Let $\{f_k(x)\}_{k\geqq 1} \cup \{f(x)\}$ be measurable functions defined on $E \in \mathscr{M}$. (Suppose that $|f(x)|, |f_k(x)| < \infty$ a.e $x \in E$.) Suppose that $f_k(x) \stackrel{\text{a.u}}{\longrightarrow} f(x)$. Prove the following statements.

(1) Show that $f_k(x) \stackrel{m}{\longrightarrow} f(x)$.

(2) Show that $f_k(x) \stackrel{\text{a.e}}{\longrightarrow} f(x)$.

36 **(Alternative Proof: Theorem 3.12 Egorov)** Show Theorem 3.12 (Egorov's Theorem) again using Extra Theorem above.

37 **(Definition 3.7)** Let $\{f_k(x)\}_{k\geqq 1}$ be measurable functions on $E \in \mathscr{M}$ and suppose that $|f_k(x)| < \infty$ a.e $x \in E$. Explain $\{f_k(x)\}_{k\geqq 1}$ is a Cauchy sequence in measure.

38 **(Theorem 3.16)** Let $\{f_k(x)\}_{k\geqq 1}$ be a Cauchy sequence in measure defined on $E \in \mathscr{M}$. Show that $\exists f(x) : |f(x)| < \infty$ a.e $x \in E$ s.t $f_k(x) \stackrel{m}{\longrightarrow} f(x)$.

39 **(Theorem 3.17 Riesz Theorem)** Let $\{f_k(x)\}_{k\geqq 1} \cup \{f(x)\}$ be a sequence of Lebesgue measurable functions defined on $E \in \mathscr{M}$. Suppose that $|f_k(x)|, |f(x)| < \infty$ a.e $x \in E$. Show that $f_k(x) \xrightarrow{m} f(x)$ if and only if $\forall \{k_l\}_{l\geqq 1}$ (a subsequence), $\exists \{k_{l_m}\}_{m\geqq 1}$ s.t $f_{k_{l_m}} \xrightarrow{\text{a.u}} f$.

40 **(Exercise 1)** Let $E \subset \mathbb{R}^d, E \in \mathscr{M}$ and let $\{f_n(x)\}_{n\geqq 1} \cup \{f(x)\}$ be measurable functions. Suppose that $f_n(x) \xrightarrow{\text{a.e}} f(x)$, $f_n(x) \xrightarrow{m} g(x)$. Prove of disprove $g(x) = f(x)$ a.e $x \in E$.

41 **(Exercise 2)** Let $f(x), f_k(x) (k \in \mathbb{N})$ be a real-valued function defined on $E \in \mathscr{M}; m(E) < \infty$. Suppose $f_k(x) > 0$ and $f_k(x) \xrightarrow{m} f(x)$. Show that $f_k^p(x) \xrightarrow{m} f^p(x), (p > 0)$. Hint. Use Theorem 3.17.

42 **(Exercise 3)** Let $\{f_k(x)\}_{k\geqq 1}$ be a sequence of measurable functions defined on $E \in \mathscr{M}$ and suppose that $f_k(x) \xrightarrow{m} 0$ on E. Let $g(x)$ be a real-valued measurable function defined on E. Suppose that $m(E) = +\infty$. Show that $f_k(x) \cdot g(x) \xrightarrow{m} 0$ is not necessarily true by giving a counter example.

43 **(Exercise 4)** Let $f_n(x) = \cos^n(x)$. Prove or disprove $f_n(x)$ converges to 0 in measure on $[0, \pi]$. Hint. $m([0, \pi]) < \infty$ so $\xrightarrow{\text{a.e}} \Leftrightarrow \xrightarrow{\text{a.u}} \Rightarrow \xrightarrow{m}$ on $[0, \pi]$.

44 **(Exercise 5)** Let $\{f_n(x)\}_{n\geqq 1}$ be a sequence of measurable function defined on $E \subset \mathbb{R}; E \in \mathscr{M}; m(E) > 0$. Suppose that $f_n(x) \xrightarrow{m} 0$. Prove of disprove

$$\lim_{n\to\infty} m\left(\{x \in E \mid |f_n(x)| > 0\ \}\right) = 0.$$

45 **(Exercise 6)** Let $E \subset \mathbb{R}, E \in \mathscr{M}$. A sequence of measurable functions $\{f_k(x)\}_{k\geqq 1}$ satisfies $f_k \geqq f_{k+1}$. Suppose that $f_k(x) \xrightarrow{m} 0$ on E. Prove of disprove $f_k(x) \xrightarrow{\text{a.e}} 0$.

46 **(Exercise 7)** Let $\{E_k\}_{k\geqq 1}$ be a sequence of Lebesgue measurable sets on $\mathbb{R}^d$. Let $f_k(x) \overset{\text{def}}{=} \chi_{E_k}(x)$.

(1) Show that $f_k(x) \xrightarrow{m} 0$ on $\mathbb{R}^d$ if and only if $m(E_k) \to 0$ as $k \to \infty$.

(2) Show that $f_k(x) \xrightarrow{\text{a.e}} 0$ on $\mathbb{R}^d$ if and only if $m\left(\limsup_{k\to\infty} E_k\right) = 0$.

47 **(Exercise 8)** Let $\{E_k\}_{k\geqq 1}$ be a sequence of Lebesgue measurable sets on $\mathbb{R}^d$. Let $f_k(x) \overset{\text{def}}{=} \chi_{E_k}(x)$. Show that $\{f_k(x)\}_{k\geqq 1}$ is a Cauchy sequence in measure if and only if $\lim_{k.j\to\infty} m(E_k \Delta E_j) = 0$.

48 **(Exercise 9)** Let $F(x), f_n(x) (n \in \mathbb{N})$ be measurable functions defined on $\mathbb{R}^1$. Suppose that $|f_n(x)| \leqq F(x)$ a.e $x \in \mathbb{R}^1$. Suppose that $\forall \epsilon > 0$, we have

$$m(\{x \in \mathbb{R}^1 \mid F(x) > \epsilon\}) < \infty.$$

Show that if $f_n(x) \xrightarrow{\text{a.e}} 0$ on $\mathbb{R}^1$ then $f_n(x) \xrightarrow{m} 0$ on $\mathbb{R}^1$. Hint. $\xrightarrow{\text{a.e}}$ if and only if $\xrightarrow{\text{a.u}}$ on a finite measure space.

49 **(Exercise 10)** Let $\{f_n(x)\}_{n \geqq 1}$ be a sequence of measurable functions on $E \in \mathscr{M}, E \subset \mathbb{R}^1$. Suppose that $f_n(x) \leqq f_{n+1}(x)$ for all $n \in \mathbb{N}$. Show that if $f_n(x) \xrightarrow{m} f(x)$ on E then $f_n(x) \xrightarrow{\text{a.e}} f(x)$ on E.

§ 3.3 Lebesgue measurable functions vs Continuous functions

(I) Lusin's Theorem

Lusin's Theorem states a relationship between a measurable function and a continuous function.

50 **(Theorem 3.18 Lusin)** Let $f(x)$ be a Lebesgue measurable function on $E \in \mathscr{M}, E \subset \mathbb{R}^n$. Suppose that $|f(x)| < \infty$ a.e $x \in E$. Show that $\forall \delta > 0$, there exists a closed set $F_\delta : m(E \setminus F_\delta) < \delta$ such that $f(x)$ is continuous on F_δ.

51 **(Corollary 3.19)** Let $f(x)$ be a measurable function defined on $E \in \mathscr{M}, E \subset \mathbb{R}^d$. Suppose that $|f(x)| < \infty$ a.e $x \in E$.

(1) Show that $\exists g(x) \in C(\mathbb{R}^d)$ (a continuous function on $\mathbb{R}^d$) s.t

$$m(\{x \in E : f(x) \neq g(x)\}) < \delta.$$

Explain that if $f(x)$ is bounded then $g(x)$ is also bounded.

(2) Suppose that E is bounded, Show that there exists $g(x) \in C(\mathbb{R}^d)$ (a continuous function on $\mathbb{R}^d$) with a compact support s.t

$$m(\{x \in E : f(x) \neq g(x)\}) < \delta.$$

52 **(Corollary 3.20)** Let $f(x)$ be a Lebesgue measurable function defined on $E \in \mathscr{M}, E \subset \mathbb{R}^d$. Suppose that $|f(x)| < \infty$ a.e $x \in E$. Show that $\exists \{g_k(x)\}_{k \geqq 1} \subset C(\mathbb{R}^d)$ (a sequence of continuous functions defined on $\mathbb{R}^d$) s.t

$$\lim_{k \to \infty} g_k(x) = f(x) \text{ a.e } x \in E.$$

53 **(Example 1)** Let $f(x)$ be a real-valued Lebesgue measurable function on $\mathbb{R}$. For all $x, y \in \mathbb{R}$, $f(x + y) = f(x) + f(y)$. Show that $f(x) \in C(\mathbb{R})$.

54 **(Exercise 1)** Let $f(x)$ be a real-valued Lebesgue measurable function on $\mathbb{R}$. Prove or disprove $\exists g(x) \in C(\mathbb{R})$ (a continuous function on $\mathbb{R}$) s.t

$$m(\{x \in \mathbb{R} \mid |f(x) - g(x)| > 0\}) = 0.$$

55 **(Exercise 2)** Let $f(x)$ be a Lebesgue measurable function defined on $[a, b]$. Show that there exists $\{P_n(x)\}_{n \geqq 1}$: a sequence of polynominal s.t

$$\lim_{n \to \infty} P_n(x) = f(x) \text{ a.e } x \in [a, b].$$

(II) measurability of composite functions

56 **(Lemma 3.21)** Let $f(x)$ be a real valued function defined on $\mathbb{R}^1$. Show that $f(x)$ is Lebesgue measurable if and only if $\forall G \in \mathscr{O}^1$ (an open set on $\mathbb{R}$), we have $f^{-1}(G) \in \mathscr{M}$.

57 **(Supplement to Lemma 3.21)** Let $f(x)$ be a real valued function defined on $\mathbb{R}^d$. Show that $f(x)$ is Lebesgue measurable if and only if $\forall B \in \mathscr{B}(\mathbb{R}^1)$ (a Borel set on $\mathbb{R}$), we have $f^{-1}(B) \in \mathscr{M}$.

58 **(Theorem 3.22)** Let $f(x) \in C(\mathbb{R})$ and let $g(x)$ be a real valued Lebesgue measurable function. Show that $h(x) = f \circ g(x)$ is a Lebesgue measurable function defined on $\mathbb{R}$.

59 **(Lemma 3.23, Corollary 3.24)** Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous transformation. Suppose $Z \subset \mathbb{R}^d, m(Z) = 0 \Rightarrow T^{-1}(Z)$ is a measure zero set. Show that $f \circ T(x)$ is a Lebesgue measurable funcntion if $f(x)$ is a real valued Lebesgue measurable function on $\mathbb{R}^d$. (Note. if T is a non-singular linear transformation, then T is continuous by Example 1 in §2.6, and $T^{-1}(Z)$ is a measure zero set for an arbitrary measure zero set Z by Theorem 2.25, 2.26.)

60 **(Exercise 1)** Let $f(x), g(x)$ be Lebesgue measurable on $\mathbb{R}$ and let $f(x) > 0$. Show that $f(x)^{g(x)}$ is Lebesgue measurable.

61 **(Exercise 2)** Let $f(x)$ be a Lebesgue measurable function on $[a, b]$ and suppose that $m \leqq f(x) \leqq M$ and $g(x)$ is monotone increasing on $[m, M]$. Show that $g \circ f(x)$ is measurable on $[a, b]$.

62 **(Exercise 3)** Let $f(x)$ be a Lebesgue measurable function on $\mathbb{R}^d$. Show that $f(x - y)$ is Lebesgue measurable on $\mathbb{R}^d \times \mathbb{R}^d$. $(= \mathbb{R}^{2d})$

63 **(Exercise 4)** Let $f(x, y)$ be a function on $\mathbb{R}^2$. Suppose that $\forall x \in \mathbb{R}$, $y \longmapsto f(x, y)$ is Lebesgue measurable and suppose that $\forall y \in \mathbb{R}$, $x \longmapsto f(x, y)$ is a continuous function. Show that $f(g(y), y)$ is a measurable function on $\mathbb{R}$ where $g(y)$ is a Lebesuge measurable function on $\mathbb{R}$.

64 **(Exercise 5)** In theorem 3.22, we show that if $g(x)$ is a real valued Lebesgue measurable function and $f(x)$ is continuous on $\mathbb{R}$, $f \circ g(x)$ is also Lebesgue measurable. However if $f(x)$ is Lebesgue measurable, $g(x) \in C(\mathbb{R})$ where $f \circ g(x)$ is not always Lebesgue measurable. Give an example.

§ 3.4 Exercise

65 **(Exercise 1)** Let I be an index set. Let $\{f_a(x) : a \in I\}$ be a family of Lebesgue measurable function. Prove or disprove $S(x) \overset{\text{def}}{=} \sup\{f_a(x) : a \in I\}$ is Lebesgue measurable.

66 **(Exercise 2)** Let $z = f(x, y)$ be a continuous function on $\mathbb{R}^2$ and let $g_1(x), g_2(x)$ be real-valued measurable functions on $[a, b] \subset \mathbb{R}$. Show that $F(x) \stackrel{\text{def}}{=} f(g_1(x), g_2(x))$ be a measurable function on $[a, b]$.

67 **(Exercise 3)** Let $f(x)$ be right-differentiable on $[a, b)$. Show that $f'_+(x)$ is measurable on $[a, b)$.

68 **(Exercise 4)** Let $f(x)$ be a measurable function defined on $E \in \mathscr{M}; E \subset \mathbb{R}^d; m(E) < \infty$ and suppose that $|f(x)| < \infty$ a.e $x \in E$. Show that $\forall \epsilon > 0$, $\exists g_\epsilon(x)$: a bounded measurable function defined on E s.t $m(\{x \in E : |f(x) - g_\epsilon(x)| > 0\}) < \epsilon$.

69 **(Exercise 5)** Let $f(x)$ and $f_n(x)$ be measurable functions defined on $A \subset \mathbb{R}, A \in \mathscr{M}$ and suppose that $|f(x)|, |f_n(x)| < \infty$ a.e $x \in A$. Suppose that $\forall \epsilon > 0$, $\exists B_\epsilon \subset A, B \in \mathscr{M} : m(A \setminus B) < \epsilon$ s.t $f_n(x) \xrightarrow{u} f(x)(x \in B)$. Show that

$$f_n(x) \xrightarrow{\text{a.e}} f(x) \text{ on } A.$$

70 **(Exercise 6)** Let $\{f_k(x)\}_{k \geqq 1}$ be a sequence of real valued measurable functions on $E \in M, E \subset \mathbb{R}$. Suppose that $m(E) < \infty$. Show that $f_n(x) \xrightarrow{\text{a.e}} 0$ on E if and only if $\forall > 0$

$$\lim_{j \to \infty} m \left(\left\{ x \in E \mid \sup_{k \geqq j} \{|f_k(x)|\} \geqq \epsilon \right\} \right) = 0.$$

71 **(Exercise 7)** Let $\{f(x)\} \cup \{f_k(x)\}_{k \geqq 1}$ be Lebesgue measurable functions defined on $[a, b]$. Suppose that $|f(x)|, |f_k(x)| < \infty$ a.e $x \in [a, b]$ and $f_k \xrightarrow{\text{a.e}} f$ on $[a, b]$. Show that there exists a sequence of Lebesgue measurable sets $\{E_n\}_{n \geqq 1} \subset \mathscr{M}$:

$$m\left([a, b] \setminus \bigcup_{n=1}^{\infty} E_n\right) = 0$$

s.t $f_k \xrightarrow{u} f$ on each E_n.

72 **(Exercise 8)** Let $\{f_k(x)\}$ be a sequence of measurable functions and suppose $f_k \xrightarrow{m} f$ on E. (Similarly suppose that $g_k \xrightarrow{m} g$.) Show that $f_k + g_k \xrightarrow{m} f + g$ on E.

73 **(Exercise 9)** Suppose that $m(E) < \infty$. Let $\{f(x)\} \cup \{f_k(x)\}_{k \geqq 1}$ be measurable functions on E. Suppose $|f(x)|, |f_k(x)| < \infty$ a.e $x \in E$. Show that $f_k(x) \xrightarrow{m} f(x)$ if and only if $\lim_{k \to \infty} \inf_{a > 0} \{a + m(\{x \in E : |f_k(x) - f(x)| > a\})\} = 0$.

74 **(Exercise 10)** Let $f_n(x)$ be a monotone increasing function defined on $[0, 1]$. (So $x < x' \Rightarrow f_n(x) \leqq f_n(x')$ holds for all $n \in \mathbb{N}$.) Suppose $f_n(x) \xrightarrow{m} f(x)$ on $[0, 1]$. Show that $\forall x_0 \in C(f)$ a continuous point of f, $f_n(x_0) \to f(x_0)(n \to \infty)$ holds.

75 **(Exercise 11)** Let $f : \mathbb{R}^d \to \mathbb{R}$ and suppose that $\forall \epsilon > 0$, $\exists G_\epsilon \subset \mathbb{R}^d; G_\epsilon \in \mathscr{O}^d, m(G_\epsilon) < \epsilon$ s.t $f(x) \in C(\mathbb{R}^d \setminus G)$. Show that $f(x)$ is a Lebesgue measurable function on $\mathbb{R}^d$.

76 **(Exercise 12)** Suppose that $f_k(x), g_k(x) \xrightarrow{m} 0$ on $E \in \mathscr{M}$. Show that $f_k(x) \cdot g_k(x) \xrightarrow{m} 0$ on E.

77 **(Exercise 13)** Let $f_k(x) \xrightarrow{m} f(x)$ on $[a,b]$. Let $g(x) \in C(\mathbb{R})$. Show that $g \circ f_k(x) \xrightarrow{m} g \circ f(x)$ on $[a,b]$. If we change $[a,b]$ to $[0,\infty)$, does the statement above still hold?

78 **(Exercise 14)** Let $E \in \mathscr{M}, E \subset \mathbb{R}^d$ and let $f(x)$ be a function defined on E (f is not necessarily a measurable function). Suppose that $\forall \delta > 0, \exists F_\delta \subset E, m(E \backslash F) < \delta$: a closed set s.t $f(x)$ is continuous on F. Show that $f(x)$ is measurable on E.

79 **(Exercise 15)** Let $\{f_n\}$ be a sequene of measurable functions on $[a,b]$. Let $f(x)$ be a real valued function on $[a,b]$ (f is not necessarily a measurable function). For all $\epsilon > 0$, we have

$$\lim_{n\to\infty} m^*(\{x \in [a,b] \mid |f_n - f| > \epsilon\}) = 0.$$

Prove of disprove $f(x)$ is a Lebesgue measurable function on $[a,b]$.

80 **(Exercise 16)** Let $f(x), f_k(x)$ be real valued measurable functions defined on $E \subset \mathbb{R}$. Suppose that $\forall \epsilon > 0$, we have

$$\lim_{j\to\infty} m\left(\bigcup_{k=j}^{\infty} \{x \mid |f_k(x) - f(x)| > \epsilon\}\right) = 0.$$

Show that $\forall \delta > 0, \exists e \subset E : m(e) < \delta$ s.t $f_k \xrightarrow{u} f$ on $E \setminus e$.

CHAPTER 4

Lebesgue Integral

§ 4.1 Lebesgue Integral: non-negative measurable functions

1 **(Definition 4.1)** Let $f(x)$ be a non negative measurable simple function on $\mathbb{R}^d$.

$$f(x) = \sum_{i=1}^{p} c_i \chi_{A_i}(x),\ \{A_i\}_{i=1}^{p} \subset \mathscr{M},\ \bigcup_{i=1}^{p} A_i = \mathbb{R}^d,\ A_i \cap A_j = \emptyset (i \neq j)$$

Suppose that $E \in \mathscr{M}$. Please define Lebesgue Integral $\int_E f(x)dx$.

2 **(Theorem 4.1)** Let $f(x), g(x)$ be non-negative measurable simple functions on $\mathbb{R}^d$ defined as below.

$$\begin{aligned} f(x) &= a_i(\text{if } x \in A_i,\ i = 1, 2 \cdots p), \\ g(x) &= b_j(\text{if } x \in B_j,\ j = 1, 2 \cdots q), \end{aligned}$$

where $\{a_i\}_{i=1}^{p} \cup \{b_j\}_{j=1}^{q} \subset [0, \infty)$, $\{A_i\}_{i=1}^{p} \cup \{B_j\}_{j=1}^{q} \subset \mathscr{M}$, and $\mathbb{R}^d = \bigcup_{i=1}^{p} A_i = \bigcup_{j=1}^{q} B_j$. Let $E \in \mathscr{M}$. Show the following properties.

(1) $\int_E cf(x)dx = c\int_E f(x)dx$.

(2) $\int_E (f(x) + g(x))dx = \int_E f(x)dx + \int_E g(x)dx$.

(3) Show that if $f(x) \leqq g(x)$, then $\int_E f(x)dx \leqq \int_E g(x)dx$.

3 **(Theorem 4.2)** Let $\{E_k\}_{k \geqq 1} \subset \mathscr{M}$ and suppose that $E_k \subset E_{k+1}$. Let $f(x)$ be a non negative simple measurable function on $\mathbb{R}^d$. Show that

$$\int_E f(x)dx = \lim_{k \to \infty} \int_{E_k} f(x)dx,\ \text{where } E = \bigcup_{k=1}^{\infty} E_k.$$

4 **(Definition 4.2)** Let $f(x)$ be a non-negative integtable function on $E \subset \mathbb{R}^d$. State the definition of $\int_E f(x)dx$. Also state the meaning of integrable function.

*

Until now, we have already defined Lesgue integral of non-negative measurable simple funtions (Definition 4.1) and that of non-negative measurable functions (Definition 4.1). However, non-negative measurable simple functions are also non-negative measurable functions, therefore, we can define its integral by Definition 4.2. So let us verify if the Definition 4.2 does not contradict to Definition 4.1.

5 **(Extra Theorem)** Show that Definition 4.1 and Definition 4.2 does not contradict for the integral of non-negative simple measurable function.

6 **(Some Properties derived from Definition 4.2)** Let $f(x), g(x)$ be non-negative measurable functions defined on $E \in \mathscr{M}$. Show the following properties with regard to integral of non-negative Lebesgue measurable functions. We will use them in proofs of the later theorems.

(1) Suppose that $f(x) \leqq g(x)$ on E. Show that $\int_E f(x)dx \leqq \int_E g(x)dx$.

(2) Show that if $f(x) \leqq g(x)$, and $g(x)$ is integrable on E, then $f(x)$ is also integrable on E.

(3) Let $A \subset E$ and $A \in \mathscr{M}$. Show that

$$\int_A f(x)dx = \int_E f(x)\chi_A(x)dx$$

(4) Show that $f(x) = 0$ a.e $x \in E$ if and only if

$$\int_E f(x)dx = 0.$$

(5) Suppose that $m(E) = 0$. Show that

$$\int_E f(x)dx = 0.$$

7 **(Theorem 4.3)** Show that if $f(x)$ is a non-negative integrable function defined on $E \in \mathscr{M}$, then $f(x)$ is finite almost everywhere on E. (i.e $m(\{x \in E \mid f(x) = \infty\}) = 0$.)

8 **(Theorem 4.4 Monotone Convergence Theorem : Beppo Levi)** Let $\{f_k(x)\}_{k \geqq 1}$ be an increasing sequence of non-negative measurable functions. (i.e $0 \leqq f_k(x) \leqq f_{k+1}(x)$.) Suppose that $\lim_{k\to\infty} f_k(x) = f(x), x \in E$. Show that

$$\lim_{k\to\infty} \int_E f_k(x)dx = \int_E f(x)dx$$

9 **(Theorem 4.5: Linearity of Lebesgue Integral)** Let $f(x), g(x)$ be non-negative measurable functions defined on E. Let α, β be non-negative constants. Show that

$$\int_E (\alpha f(x) + \beta g(x))dx = \alpha \int_E f(x)dx + \beta \int_E g(x)dx.$$

10 **(Example 2)** Let $\{f_k(x)\}_{k\geqq 1}$ be a decreasing sequence of non-negative integrable functions. Suppose that $\lim_{k\to\infty} f_k(x) = f(x)$ for all $x \in E$. Show that

$$\lim_{k\to\infty}\int_E f_k(x)dx = \int_E f(x)dx.$$

11 **(Example 3)** Let $f(x), g(x)$ be non-negative measurable functions defined on E. Suppose $f(x) = g(x)$ a.e $x \in E$. Show that

$$\int_E f(x)dx = \int_E g(x)dx$$

12 **(Supplement to Theorem 4.5 and Example 2)** Show that the assumption $f_k(x) \to f(x)$ for all $x \in E$ can be modified to $f_k(x) \xrightarrow{\text{a.e}} f(x)$ on E in Theorem 4.5 (and Example 2).

13 **(Exercise 1)** Let $f_1, f_2, \cdots, f_m$ be a non-negative integrable function on E. Show the following statements.

(1) $F(x) = \left(\sum_{i=1}^m (f_i(x))^2\right)^{1/2}$ is integrable on E.

(2) $G(x) = \sum\sum_{1\leqq i,k\leqq m} (f_i(x)f_k(x))^{1/2}$ is integrable on E.

14 **(Exercise 2)** Let $\{E_k\}_{k\geqq 1}$ be an increasing sequence of point sets on $\mathbb{R}^d$. Suppose that $E_k \nearrow E$ as $k \to \infty$. If $f(x)$ is non-negative measurable on E, show that

$$\int_E f(x)dx = \lim_{k\to\infty}\int_{E_k} f(x)dx.$$

15 **(Exercise 3)** Let $\{f_k\}_{k\geqq 1}$ be a sequence of non-negative measurable functions defined on E. Suppose that $\lim_{k\to\infty}\int_E f_k(x)dx = 0$. Show that

$$\lim_{k\to\infty}\int_E (1-\exp(-f_k(x)))\,dx = 0.$$

Hint. $1 - e^{-t} \leqq t$ when t is non-negative.

16 **(Exercise 4)** Let $f(x)$ be a non-negative integrable funtion defined on E. Show that for any $\epsilon > 0$, there exists $N > 0$ s.t.

$$\int_E f(x)\chi_{\{x\in E\ \mid\ f(x)>N\}}(x)dx < \epsilon.$$

17 **(Exercise 5)** Show that

$$\lim_{n\to\infty}\int_{[0,n]}\left(1+\frac{x}{n}\right)^n \exp(-2x)\,dx = \int_{[0,\infty)} \exp(-x)\,dx.$$

18 **(Exercise 6)** Show that

$$\lim_{n\to\infty}\int_{[0,1]} x^n dx = 0.$$

19 **(Theorem 4.6 Swap Σ and $\int$)** Let $\{f_k(x)\}_{k\geqq 1}$ be a sequence of non-negative measurable functions defined on $E \in \mathscr{M}$. Show that

$$\int_E \sum_{k=1}^{\infty} f_k(x)dx = \sum_{k=1}^{\infty} \int_E f_k(x)dx$$

20 **(Corollary 4.7)** Let $E_k \in \mathscr{M}(k = 1, 2, \cdots)$ and suppose that $E_i \cap E_j = \emptyset (i \neq j)$. Let $f(x)$ be a non-negative measurable function defined on $E \stackrel{\text{def}}{=} \bigcup_{k=1}^{\infty} E_k$. Show that

$$\int_E f(x)dx = \sum_{k\geqq 1} \int_{E_k} f(x)dx$$

21 **(Example 4)** Suppose that $E_1 E_2 \cdots E_n \in \mathscr{M}([0,1])$ and suppose that $\forall x \in [0,1]$, $\#\{i = 1, 2 \cdots n \mid x \in E_i\} \geqq k$. $(k \leqq n)$ Show that there exists $E_{i_0} (i_0 = 1, 2 \cdots n)$ s.t $m(E_{i_0}) \geqq \frac{k}{n}$.

22 **(Theorem 4.8 Fatou's Lemma)** Let $\{f_k\}_{k\geqq 1}$ be non-negative measurable functions on $E \in \mathscr{M}$. Show that

$$\int_E \liminf_{k\to\infty} f_k(x)dx \leqq \liminf_{k\to\infty} \int_E f_k(x)dx$$

23 **(Example 5: equality does not always hold in Fatou's lemma)** Consider a sequene of non-negative measurable functions on $[0,1]$. Does equality hold for the Fatou's lemma?

$$f_n(x) = \begin{cases} 0 & x = 0 \\ n & 0 < x < 1/n \\ 0 & 1/n \leqq x \leqq 1 \end{cases}$$

24 **(Theorem 4.9)** Let $f(x)$ be a non-negative measurable function on $E \in \mathscr{M}, m(E) < \infty$ and suppose that $|f(x)| < \infty$ a.e $x \in E$. In $[0, \infty)$, we consider a segmentation as below.

$$0 = y_0 < y_1 < \cdots < y_k < y_{k+1} < \cdots \to \infty$$

We suppose that $y_{k+1} - y_k < \delta$. We define E_k as below.

$$E_k \stackrel{\text{def}}{=} \{x \in E \mid y_k \leqq f(x) < y_{k+1}\} \ (k = 0, 1, 2 \cdots)$$

(1) Show that $f(x)$ is integrable on E if and only if

$$\sum_{k=0}^{\infty} y_k m(E_k) < \infty$$

(2) Show that

$$\lim_{\delta \searrow 0} \sum_{k=0}^{\infty} y_k m(E_k) = \int_E f(x)dx.$$

*

In the question above, you may feel that the limit on the left hand side is somewhat weird because partition $\{y_k\}_{k=0}^{\infty}$ is not unique. Let

$$P^{(\delta)} \overset{\text{def}}{=} \{\{y_k\}_{k=0}^{\infty} \mid y_0 = 0 < y_1 \cdots < y_n \nearrow \infty \ ; \ y_k - y_{k-1} < \delta, \forall k \in \mathbb{N}\}.$$

And for each partition $I \in P^{(\delta)}$, we define

$$S(I) \overset{\text{def}}{=} \sum_{k=0}^{\infty} y_k m(E_k), I = \{y_k\}_{k=0}^{\infty}.$$

Note that if $\delta < \delta'$, then

$$\sup_{I \in P^{(\delta)}} S(I) \leqq \sup_{I \in P^{(\delta')}} S(I), \quad \inf_{I \in P^{(\delta)}} S(I) \geqq \inf_{I \in P^{(\delta')}} S(I).$$

So $\lim_{\delta \searrow 0} \sup_{I \in P^{(\delta)}} S(I)$ and $\lim_{\delta \searrow 0} \inf_{I \in P^{(\delta)}} S(I)$ exist. We need to prove that

$$\int_E f(x)dx = \lim_{\delta \searrow 0} \sup_{I \in P^{(\delta)}} S(I) = \lim_{\delta \searrow 0} \inf_{I \in P^{(\delta)}} S(I).$$

25 **(Example 6)** Let $E \in \mathscr{M}, (E \subset \mathbb{R}) : m(E) < \infty$. Let $f(x)$ be a non-negative real-valued measurable function on E. (i.e $f(x) : E \mapsto [0, \infty)$) Show that $f(x)$ is integrable on $[0, \infty]$ if and only if

$$\sum_{k=0}^{\infty} m\left(\{x \in E | \ f(x) \geqq k\right) < \infty$$

26 **(Example 7)** Let $f(x) : [a, b] \mapsto [0, \infty)$ be a non-negative real-valued measurable function. Show that $f(x)^2$ is integrable on $[a, b]$ if and only if

$$\sum_{n=1}^{\infty} nm\left(\{x \in [a, b] | \ f(x) \geqq n\}\right) < \infty.$$

27 **(Exercise 7)** Let $f(x)^3$ be a non-negative integrable function on $E \in \mathscr{M}, m(E) < \infty$. Show that $f(x)^2$ is also integrable on E.

28 **(Exercise 8)** Let $f(x) : [a, b] \mapsto [0.\infty)$ be a non-negative real-valued measurable function on $[a, b]$. Show that $f(x)^3$ is integrable on $[a, b]$ if and only if

$$\sum_{n=1}^{\infty} n^2 m\left(x \in [a, b] | \ f(x) \geqq n\right)$$

29 **(Exercise 9)** Let $\{f_k\}_{k \geqq 1}$ be a sequence of non-negative measurable functions on $E \in \mathscr{M}$. Suppose that $\lim_{k \to \infty} f_k(x) = f(x), \ f_k(x) \leqq f(x)$. Show that for any $e \subset E, e \in \mathscr{M}$, we have

$$\lim_{k \to \infty} \int_e f_k(x)dx = \int_e f(x)dx$$

30 **(Exercise 10)** Let $\{E_n\} \subset [0, 1]$ be a sequence of Lebesgue measurable sets. Suppose that $m\left(\limsup E_n\right) = 0$. Show that $\forall \epsilon > 0, \exists A \subset [0, 1]; \ A \in \mathscr{M}; \ m\left([0, 1] \setminus A\right) < \epsilon$ s.t

$$\sum_{n=1}^{\infty} m\left(A \cap E_n\right) < \infty$$

§ 4.2 Lebesgue Integral: general measurable functions

(I) Definition of Integral and Basic Properties

In the last section, we defined integral of non-negative measurable functions. From now on, we study integral of general (not necessarily non-negative) measurable functions.

31 **(Definition: integral of general measurable functions)** Let $f(x)$ be a measurable function on $E \in \mathbb{R}^d; E \in \mathscr{M}$.

(1) Define $\int_E f(x)dx$. Explain the meaning of $\int_E f(x)dx$ exists.

(2) Explain the meaning of $f(x)$ is integrable.

(3) Explain that $f(x)$ is integrable if and only if $|f(x)|$ is integrable.

(4) Explain that $\left|\int_E f(x)dx\right| \leqq \int_E |f(x)|\,dx$

*

From now on, let $L(E)$ be a set of all integrable functions defined on $E \in \mathscr{M}$.

$$L(E) \overset{\text{def}}{=} \left\{ f(x) : \text{measurable} \;\mid\; \int_E |f(x)| < \infty \right\}$$

32 **(Example 1)** Let $f(x)$ be a bounded function on $E \in \mathscr{M}$ and suppose that $m(E) < \infty$. Is $f(x)$ integrable on E?

33 **(Some Properties)** Show the following properties.

(1) Suppose that $f(x) \in L(E)$. Show that $|f(x)| < \infty$ a.e $x \in E$.

(2) Let $E \in \mathscr{M}$. Suppose that $f(x) = 0$ a.e $x \in E$. Show that $\int_E f(x)dx = 0$.

(3) Let $f(x)$ be a measurable function on E. Let $g \in L(E)$. Suppose that $|f(x)| \leqq g(x)$. Show that $f(x) \in L(E)$.

(4) Let $f(x) \in Ł(\mathbb{R}^d)$. Show that

$$\lim_{N\to\infty} \int_{\{x\in\mathbb{R}^d \mid\ |x|\geqq N\}} |f(x)|dx = 0$$

34 **(Theorem 4.10 Linearity of Lebesgue Integral)** Let $E \in \mathscr{M}$. Suppose that $f(x) \in L(E)$ and $\int_E g(x)dx$ exists ($g(x)$ is not necessarily integrable), and let $C \in \mathbb{R}$.

(1) Show that

$$\int_E Cf(x)dx = C\int_E f(x)dx$$

(2) Show that

$$\int_E (f(x) + g(x))dx = \int_E f(x)dx + \int_E g(x)dx$$

35 **(Example 2)** Let $f(x)$ be a measurable function on $[0,1]$. Show that $f \in L([0,1])$ if the following statement holds.

$$\int_{[0,1]} |f(x)| \ln(1 + |f(x)|)\, dx < \infty.$$

36 **(Example 3)** Let $\{f_n(x)\} \subset L(E)$ and suppose that $\lim_{n\to\infty} f_n(x) = f(x) (\forall x \in E)$ and $f_n(x) \leqq f_{n+1}(x) (\forall n \in \mathbb{N}, \forall x \in E)$. Show that

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx$$

37 **(Example 4)** Let $g(x) \in L(E)$ and let $\{f_n(x)\}_{n\geqq 1} \subset L(E)$. Suppose that $f_n(x) \geqq g(x)$ a.e $x \in E$. Show that

$$\int_E \liminf_{n\to\infty} f_n(x)dx \leqq \liminf_{n\to\infty} \int_E f_n(x)dx$$

38 **(Example 5 Jenesen's inequality)** Let $w(x)$ be a positive-valued measurable function on $E \subset \mathbb{R}; E \in \mathscr{M}$ and suppose that $\int_E w(x)dx = 1$. Let $\phi(x)$ be a convex function on $I = [a,b]$. Let $f(x)$ be a measurable function on E and suppose that $R(f) \subset I$. Show that if $f(x) \cdot w(x) \in L(E)$ then we have

$$\phi\left(\int_E f(x) \cdot w(x)dx\right) \leqq \int_E \phi \circ f(x) w(x)dx.$$

39 **(Exercise 1)** Let $f(x), g(x) \in L(\mathbb{R}^d)$. Show that $\min\{f(x), g(x)\}, \max\{f(x), g(x)\}$ are integrable.

40 **(Exercise 2)** Let $f(x,y) : [0,1]^2 \mapsto \mathbb{R}$

$$f(x) = \begin{cases} 1 & xy \notin \mathbb{Q} \\ 2 & xy \in \mathbb{Q} \end{cases}$$

Show that

$$\int\int_{[0,1]\times[0,1]} f(x,y)dxdy = 1.$$

41 **(Exercise 3)** Let $f(x) \in L(E)$. Show that

$$m(\{x \in E : |f(x)| > k\}) = o\left(\frac{1}{k}\right).$$

$(\cdots) = o(\frac{1}{k})$ means that the left hand side converges to 0 faster than $\frac{1}{k}$ as $k \to \infty$.

42 **(Exercise 4)** Let $f(x) \in L((0,\infty))$. Let $f_n(x) \stackrel{\text{def}}{=} f(x)\chi_{(0,n)}(x)$. Show that $f_n(x) \stackrel{m}{\longrightarrow} f(x)$ on $(0,\infty)$.

43 **(Exercise 5)** Let $f(x) \in L([0,1])$ and suppose that $\exp\left(\int_{[0,1]} f(x)dx\right) = \int_{[0,1]} \exp(f(x))\, dx$. Show that there exists C : a constant s.t $f(x) = C$ a.e $x \in [0,1]$. Hint. $e^C(x-C) + e^C \leqq e^x$. Equality holds if $x = C$.

44 **(Exercise 6)** Let $f(x) \in L(\mathbb{R}^1)$. For $\forall I$: a bounded interval, we define $f_I \stackrel{\text{def}}{=} \frac{1}{|I|}\int_I f(x)dx$ and $E_I \stackrel{\text{def}}{=} \{x \in I : f(x) > f_I\}$. Show that

$$\int_I |f(x) - f_I| dx = 2\int_{E_I} (f(x) - f_I)\, dx$$

45 **(Therorem 4.11: countable additivity about range)** Let $E_k \in \mathscr{M}$ and suppose that $E_i \cap E_j = \emptyset$ if $i \neq j$. Let $f(x)$ be a measurable function on $E \stackrel{\text{def}}{=} \cup_{k=1}^{\infty} E_k$. Suppose $\int_E f(x)dx$ exists. Show that

$$\int_E f(x)dx = \sum_{k=1}^{\infty} \int_{E_k} f(x)dx.$$

46 **(Example 6: test condition to be 0 almost everywhere)** Let $f(x) \in L([a,b])$. Show that if for $\forall c \in [a,b]$, $\int_{[a,c]} f(x)dx = 0$ then,

$$f(x) = 0 \text{ a.e } x \in [a,b]$$

47 **(Example 7)** Let $g(x) : E \mapsto \mathbb{R}$ be a real-valued measurable function on $E \in \mathscr{M}$. Suppose that $\forall f(x) \in L(E)$, $f(x)g(x) \in L(E)$. Show that $\exists Z \in \mathscr{M}$ with $m(Z) = 0$ s.t $g(x)$ is bounded on $E \setminus Z$.

48 **(Therorem 4.12: absolute continuity of integral)** Let $f(x) \in L(E)$. Show that $\forall \epsilon > 0, \exists \delta > 0$ s.t $\forall e \in \mathscr{M}(e \subset E)$ with $m(e) < \delta$, the following inequality holds.

$$\left|\int_e f(x)dx\right| \leqq \int_e |f(x)|dx < \epsilon.$$

49 **(Example 8)** Let $f : E \mapsto [0,\infty]$, $f(x) \in L(E)$, $E \subset \mathbb{R}; E \in \mathscr{M}$. Suppose that $0 < A = \int_E f(x)dx < \infty$. Show that there exists $e \in \mathscr{M}; e \subset E$ s.t

$$\int_e f(x)dx = \frac{A}{3}.$$

50 **(Therorem 4.13: translation of variables in Lebesgue Integral)** Suppose that $\int_{\mathbb{R}^d} f(x)dx$ exists and let $y_0 \in \mathbb{R}^d$. Show that $f(x+y_0) \in L(E)$ and

$$\int_{\mathbb{R}^d} f(x+y_0)dx = \int_{\mathbb{R}^d} f(x)dx.$$

.

51 **(Example 9)** Let $f(x) \in L(E), E \stackrel{\text{def}}{=} [0,\infty]$. Show that

$$\lim_{n\to\infty} f(x+n) = 0 \text{ a.e } x \in E.$$

Hint. It is enough to show that $\lim_{n\to\infty} f(x+n) = 0$ a.e $x \in [0,1)$. You may consider $\sum_{n=0}^{\infty}\int_{[0,1)} |f(x+n)|dx$.

52 **(Example 10)** Let $I \subset \mathbb{R}$ be an interval and let $\int_I f(x)dx$ exists. For $a \neq 0$, we define $J \overset{\text{def}}{=} \{\frac{x}{a} : x \in I\}$ and $g(x) \overset{\text{def}}{=} f(ax), x \in J$. Show that $\int_J g(x)dx$ exists and

$$\int_I f(x)dx = |a| \int_J g(x)dx.$$

53 **(Exercise 7)** Let $f(x), g(x) \in L(\mathbb{R})$ and $\int_{[a,x]} f(t)dt = \int_{[a,x]} g(t)dt$ for all $x \in \mathbb{R}$. Show that $f(x) = g(x)$ a.e $x \in [a, \infty)$.

54 **(Exercise 8)** Let $f(x) \in L(\mathbb{R})$. Let ϕ be an arbitrary bounded Lebesgue measurable function. Suppose that $\int_{\mathbb{R}} f(x)\phi(x)dx = 0$. Show that $f(x) = 0$ a.e $x \in \mathbb{R}$.

(II) Lebesgue Dominated Convergence Theorem

55 **(Therorem 4.14: Lebesgue Dominated Convergence Theorem (L.D.C.T))** Let $\{f_k(x)\}_{k\geqq 1}$ be a sequence of measurable functions on $E \in \mathscr{M}$. Suppose that $\lim_{k\to\infty} f_k(x) = f(x)$ a.e $x \in E$ and suppose that for every $k \in \mathbb{N}$, $|f_k(x)| \leqq g(x)$ a.e $x \in E$ where $g \in L(E)$. Show that

$$\lim_{k\to\infty} \int_E f_k(x)dx = \int_E f(x)dx.$$

Hint. You can try to show that $\limsup_{n\to\infty} \int_E |f_n(x) - f(x)|dx = 0$.

56 **(Therorem 4.15 L.D.C.T convergence in measure version)** Let $\{f_k(x)\}_{k\geqq 1}$ be a sequence of measurable functions defined on $E \subset \mathbb{R}^d, E \in \mathscr{M}$ and suppose that $f_k(x) \xrightarrow{m} f(x)$ on E. We also suppose that $\exists g(x) \in L(E)$ s.t $|f_k(x)| \leqq g(x)$ a.e $x \in E$. Show that $f(x) \in L(E)$ and

$$\lim_{k\to\infty} \int_E f_k(x)dx = \int_E f(x)dx$$

57 **(Example 12)** Show that

$$\int_{[0,1]} \frac{x\sin(x)}{1+(nx)^\alpha}dx = o\left(\frac{1}{n}\right) \ (n\to\infty, \alpha > 1).$$

58 **(Example 13)** Show that

$$\int_{[\alpha,\infty)} \frac{x\exp\left(-n^2x^2\right)}{1+x^2} = o\left(\frac{1}{n^2}\right) (n\to\infty, \alpha > 0)$$

59 **(Exercise 1)** Let $f(x), F(x), \phi(x), \phi_n(x)$ be Lebesgue measurable functions defined on $[a,b]$. Suppose that $\phi_n(x) \to \phi(x)$ for all $x \in [a,b]$ and $|f(x)\phi_n(x)| \leqq F(x) \in L([a,b])$. We also suppose that

$$\int_{[a,x]} f(t)\phi_n(t)dt = \phi_n(x) - \phi_n(a)\ \forall x \in [a,b],$$

Show that
$$\int_{[a,x]} f(t)\phi(t)dt = \phi(x) - \phi(a)\ \forall x \in [a,b].$$

60 **(Exercise 2)** Show that

$$\cos(nx) \overset{m}{\not\to} 0 \text{ on } [-\pi,\pi).$$

Hint. You may use the fact that $\int_{[-\pi,\pi)} \cos(2nx)dx = 0$ without proof. We will study the relationship between Lebesgue integral and Riemann integral in the following section.

61 **(Exercise 3)** Let $f \in L((0,\infty))$. Show that $g(x)$ is continuous on $(0,\infty)$.

$$g(x) \overset{\text{def}}{=} \int_{(0,\infty)} \frac{f(t)}{x+t}dt$$

62 **(Exercise 4)** Let $f \in L(E)$ and let $E_k \overset{\text{def}}{=} \{x \in E : |f(x)| < 1/k\}$. Show that

$$\lim_{k\to\infty} \int_{E_k} |f(x)|\, dx = 0.$$

63 **(Exercise 5)** Let $\{f_k\} \cup \{g_k\} \cup \{f, g\} \subset L(E)$. Suppose that $|f_k(x)| \leqq M < \infty$ and $\int_E |f_k(x) - f(x)|\, dx \to 0$, $\int_E |g_k(x) - g(x)|\, dx \to 0$ as $k \to \infty$. Show that

$$\int_E |f_k(x)g_k(x) - f(x)g(x)|\, dx \to 0 \text{ as } k \to \infty.$$

64 **(Exercise 6)** Let $\{f_k(x)\} \subset L(E)$ and suppose that $f_k \overset{u}{\to} f$ on $E \in \mathscr{M}; m(E) < \infty$. Show that

$$\lim_{k\to\infty} \int_E f_k(x)dx = \int_E f(x)dx.$$

65 **(Corollary 4.16)** Let $f_k(x) \in L(E), k = 1, 2\cdots$. Suppose that $\sum_{k=1}^{\infty} \int_E |f_k(x)| < \infty$.

(1) Show that

$$\sum_{k=1}^{\infty} f_k(x) \text{ converges a.e } x \in E.$$

(i.e $\sum_{k=1}^{\infty} f_k(x)$ exists and is finite a.e $x \in E$.)

(2) Show that

$$\sum_{k=1}^{\infty} \int_E f_k(x)dx = \int_E S(x)dx$$

where $S(x) = \sum_{k=1}^{\infty} f_k(x)$. $S(x)$ is a measurable function defined a.e $x \in E$, but is not defined at every $x \in E$. However, we can still regard $S(x)$ as a measurable function defined on E because it does not have influence on its integral. If you feel weird, you can also define $S(x)$ in the following way instead. Then $S(x)$ is defined at every $x \in E$ and is a measurable function on E.

$$S(x) \overset{\text{def}}{=} \begin{cases} \sum_{k=1}^{\infty} f_k(x) & \text{if converges} \\ 0 & \text{otherwise} \end{cases}$$

This operation is called a measurable modification.

66 **(Therorem 4.17 integral and differentiation)** Let $f(x,y)$ be a function defined on $E \times (a,b)$. Suppose that $f(x,y)$ as a function of x under fixed y

$$f_{|y} : x \mapsto f(x,y)$$

is integrable on E for all $y \in (a,b)$, and also suppose that $f(x,y)$ as a function of y under fixed x

$$f_{|x} : y \mapsto f(x,y)$$

is differentiable respect to y for all $x \in E$. Suppose that $\exists F(x) \in L(E)$ s.t $\left|\frac{\partial}{\partial y} f(x,y)\right| \leqq F(x)$ for all $(x,y) \in E \times (a,b)$. Show that

$$\frac{\partial}{\partial y} \int_E f(x,y)dx = \int_E \frac{\partial}{\partial y} f(x,y)dx$$

67 **(Example 14)** Let $f(x), f_n(x)$ be integrable and real-valued on $\mathbb{R}$. Suppose $\forall E \in \mathscr{M}; E \subset \mathbb{R}$, $\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx$. Show that

$$\liminf_{n\to\infty} f_n(x) \leqq f(x) \leqq \limsup_{n\to\infty} f_n(x)$$

68 **(Exercise 7)** Let $f(x)$ be non-negative and integrable on $[0,\infty)$ and let $E \subset (0,\infty)$. Suppose that $\int_E f(x)dx = 1$. Show that

$$\int_E f(x)\cos(x)dx \neq 1.$$

69 **(Exercise 8)** Let $f(x), f_n(x) \in L(\mathbb{R})$ and suppose that $\int_{\mathbb{R}} |f_n(x) - f(x)|\, dx \leqq \frac{1}{n^2}$. Show that

$$f_n(x) \to f(x) \text{ a.e } x \in \mathbb{R}.$$

70 **(Exercise 9)** Let $\{a_n\}$ be a sequence of real numbers and suppose that $|a_n| < \ln(n)$. Show that

$$\int_{[2,\infty)} \sum_{n=2}^{\infty} a_n n^{-x} dx = \sum_{n=2}^{\infty} \frac{a_n}{\log n} n^{-2}.$$

We still do not know the relationship between Lebesgue integral and Rieman improper integral. In this question, you may suppose that $\int_{[2,\infty)} n^{-x}dx = \frac{1}{n^2 \log n}$.

71 **(Exercise 10)** Let $f(x,y)$ be a function defined on $E \times \mathbb{R}^d$. Suppose that $\forall y \in \mathbb{R}^d, f(x,y)$ is a Lebesgue measurable function on E and suppose that $\forall x \in E$, $f(x,y)$ is a continuous function on $\mathbb{R}^d$. Moreover suppose that $\exists g \in L(E)$ s.t $|f(x,y)| \leqq g(x)$ a.e $x \in E$. Show that F is continuous on $\mathbb{R}^d$.

$$F(y) \overset{\text{def}}{=} \int_E f(x,y)dx.$$

§ 4.3 Integrable functions vs Continuous functions

72 **(Theorem 4.18)** Let $f \in L(E); E \subset \mathbb{R}^n$. Show that $\forall \epsilon > 0$, $\exists g(x) \in C(\mathbb{R}^n)$ with a compact support s.t

$$\int_E |f(x) - g(x)| \, dx < \epsilon.$$

73 **(Corollary 4.19; 4.20)** Let $f \in L(E)$. Show that there exists $\{g_k(x)\} \subset C(\mathbb{R}^n)$ with a compact support s.t

$$(i) \quad \lim_{k\to\infty} \int_E |f(x) - g_k(x)| \, dx = 0;$$
$$(ii) \quad \lim_{k\to\infty} g_k(x) = f(x) \text{ a.e } x \in E.$$

74 **(Example 1)** Let $f \in L(\mathbb{R}^n)$. Suppose that $\forall \phi(x) \in C(\mathbb{R}^n)$ with a compact support we have

$$\int_{\mathbb{R}^n} f(x)\phi(x)dx = 0.$$

Show that

$$f(x) = 0 \text{ a.e } x \in \mathbb{R}^n$$

75 **(Theorem 4.21 Mean Continuity)** Let $f \in L(\mathbb{R}^n)$. Show that

$$\lim_{x_0\to 0} \int_{\mathbb{R}^n} |f(x + x_0) - f(x)| \, dx = 0.$$

76 **(Example 3)** Let $E \in \mathscr{M}; E \subset \mathbb{R}^n$. Show that

$$\lim_{|h|\to 0} m\left(E \cap (E + \{h\})\right)$$

77 **(Corollary 4.22)** Let $f \in L(E)$. Show that we may find a sequence of step functions $\{\phi_k(x)\}$ s.t

$$(i) \quad \lim_{k\to\infty} \phi_k(x) = f(x) \text{ a.e } x \in E,$$
$$(ii) \quad \lim_{k\to\infty} \int_E |f(x) - \phi_k(x)| \, dx = 0.$$

78 **(Example 4: Extension of Riemann Lebesgue's Lemma)** Suppose that $\{g_n(x)\}$ is a sequence of Lebesgue measurable functions defined on $[a, b]$ which satisfies the following two conditions.

$$(i) \quad |g_n(x)| \leqq M \ (x \in [a, b])$$
$$(ii) \quad \forall c \in [a, b], \ \lim_{n\to\infty} \int_{[a,c]} g_n(x)dx = 0.$$

Show that $\forall f \in L([a, b])$, we have

$$\lim_{n\to\infty} \int_{[a,b]} f(x)g_n(x)dx = 0.$$

79 **(Example 5)** Let $\{\lambda_n\}$ be a sequence of real numbers. Suppose $\lambda_n \to \infty$ as $n \to \infty$. Show that A is a measure zero set.

$$A \overset{\text{def}}{=} \left\{ x \in \mathbb{R} \mid \lim_{n\to\infty} \sin(\lambda_n x) \text{ exists.} \right\}.$$

You may use the fact that $\int_a^b \sin \lambda_n x dx = -(\cos(b \cdot \lambda_n) - \cos(a \cdot \lambda_n))$ without proof.

80 **(Example 6)** Let $f(x)$ be a bounded measurable function defined one $[0,1]$. Suppose that

$$I_n = \int_{[0,1]} x^n f(x)dx = 0 \ (n = 1, 2 \cdots).$$

Show that $f(x) = 0$ a.e $x \in [0,1]$.

81 **(Example 7)** Let $f(x)$ be a non-negative measurable function on $\mathbb{R}$. Show that there exists an increasing sequence of closed sets $\{F_n\}_{n \geqq 1}$ s.t

$$m\left(\mathbb{R} \setminus \cup_{n\geqq 1}^{\infty} F_n\right) = 0, \ f(x) \in C(F_n).$$

§ 4.4 Lebesgue Integral vs Riemann Integral

82 **(Darboux Theorem)** Let $f(x)$ be a bounded function defined on $I = [a,b]$. We consider Riemann Integral of $f(x)$ on $I = [a,b]$. We denote it as (R) $\int_a^b f(x)dx$ to distinguish from Lebesgue integral (L) $\int_{[a,b]} f(x)dx$.

(1) Let $\Delta \overset{\text{def}}{=} \{x_0, x_1 \cdots x_n\}$ be a partition of the interval $[a,b]$. ($a = x_0 < x_1 < \cdots < x_k = b$.) Let $\overline{S}(\Delta) \overset{\text{def}}{=} \sum_{i=1}^k \sup_{x\in[x_{i-1},x_i]}\{f(x)\}(x_i - x_{i-1})$ and let $\underline{S}(\Delta) \overset{\text{def}}{=} \sum_{i=1}^k \inf_{x\in[x_{i-1},x_i]}\{f(x)\}(x_i - x_{i-1})$. Define $\overline{\int_a^b} f(x)dx$ and $\underline{\int_a^b} f(x)dx$ using $\overline{S}(\Delta)$ and $\underline{S}(\Delta)$.

(2) $\forall \Delta_1, \Delta_2$, $\underline{S}(\Delta_1) \leqq \overline{S}(\Delta_2)$ ($\because \underline{S}(\Delta_1) \leqq \underline{S}(\Delta_1 \cup \Delta_2) \leqq \overline{S}(\Delta_1 \cup \Delta_2) \leqq \overline{S}(\Delta_2)$), so we have $\underline{\int_a^b} f(x)dx \leqq \overline{\int_a^b} f(x)dx$. Let $|\Delta| \overset{\text{def}}{=} \max\{x_i - x_{i-1}\}_{i=1}^k$. Let us consider a sequence of partition $\{\Delta_n\}_{n\geq 1}$ s.t $|\Delta_n| \searrow 0$. Show that

$$\overline{S}(\Delta_n) \to \overline{\int_a^b} f(x)dx, \ \underline{S}(\Delta_n) \to \underline{\int_a^b} f(x)dx.$$

A sequence of partition $\{\Delta_n\}_{n\geq 1}$ with $|\Delta_n| \searrow 0$ is not unique. However this theorem assures that $\overline{S}(\Delta_n) \to \overline{\int_a^b} f(x)dx$ and $\underline{S}(\Delta_n) \to \underline{\int_a^b} f(x)dx$ for any sequence with $|\Delta_n| \searrow 0$. Therefore it is enough for us to give an arbitrary $\{\Delta_n\}_{n\geq 1}$ with $|\Delta_n| \searrow 0$ in proofs of later lemmas and theorems.

(3) Explain the meaning of Riemann Integrable.

83 **(Lemma 4.23)** Let $f(x)$ be a bounded function defined on $[a,b]$ and let $S(f;[a,b])$. Let $\omega_f(x_0) \overset{\text{def}}{=} \lim_{\delta\searrow 0}\sup_{x',x''\in B(x_0,\delta)}\{|f(x')-f(x'')|\}$. (We have defined this function in Chapter1.) Show that

$$\text{(L)}\int_{[a,b]}\omega_f(x)dx = \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx.$$

(L) means that the integral is Lebesgue integral. (R) means that the integral is Riemann integral. We sometimes add (L) or (R) before $\int$ to clarify whether the integral is Lebesgue integral or Riemann integral.

84 **(Theorem 4.24)** Let $f(x)$ be a bounded function on $[a,b]$. Show that $f(x)$ is Riemann-integrable if and only if

$$m\left(\{x\in[a,b] \mid f \text{ is discontinuous at } x\}\right) = 0.$$

85 **(Theorem 4.25)** Let $f(x)$ be a bounded function on $[a,b]$. Show that if $f(x)$ is Riemann integrable on $I=[a,b]$, $f(x)$ is Lebesgue measurable, Lebesgue integrable and

$$\text{(R)}\int_a^b f(x)dx = \text{(L)}\int_I f(x)dx.$$

We may say that Lebesgue integral is an extension of Riemann integral. (1. However Lebesgue integrability does not imply Riemann integrability. 2. Riemann improper integral exists does not imply Lebesgue integrable. We consider an integral of a bounded function defined on a bounded interval now.)

86 **(Exercise 1)** Let $F\subset[0,1]$ be a closed set and suppose that $m(F)=0$. Show that $\chi_F(x)$ is Riemann integrable on $[0,1]$.

87 **(Exercise 2)** Let $f:[0,1]\to[a,b]$ is a Riemann integrable function and let $g\in C([a,b])$. Show that $g\circ f$ is Riemann integrable on $[0,1]$.

88 **(Exercise 3)** Let f,g be Riemann integrable functions on $[a,b]$ and let $E\subset[a,b], \overline{E}=[a,b]$. Suppose that $f(x)=g(x), \forall x\in E$. Show that

$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

89 **(Theorem 4.26)** Let $\{E_k\}\subset\mathscr{M}$ be an increasing sequence of Lebesgue measurable sets. Let $E\overset{\text{def}}{=}\bigcup_{k\geq1}E_k$. Suppose that $f\in L(E_k), k=1,2\cdots$ and suppose that $\lim_{k\to\infty}\int_{E_k}|f(x)|\,dx$ exists and is finite (converges). Show that $f(x)\in L(E)$ and

$$\int_E f(x)dx = \lim_{k\to\infty}\int_{E_k} f(x)dx.$$

Hint. We can easily prove this theorem by using monotone convergence theorem and Lebesgue Dominated Convergence Theorem. However, this theorem teaches us a relationship between Riemann improper integral and Lebesgue integral. Suppose that $f(x)$ is Riemann integrable on $[0,k]$ for each $k\in\mathbb{N}$ and

$\lim_{k\to\infty}$ (R) $\int_{[0,k]} |f(x)|\, dx < \infty$. (In otherwords the Riemann improper integral converges absolutely.) Then we have the following conclusion. First, since Riemann integrability implies Lebesgue integrability, we have

$$(\mathrm{R}) \int_{[0,k]} |f(x)|\, dx = (\mathrm{L}) \int_{[0,k]} |f(x)|\, dx.$$

Second, by monotone convergence theorem we have

$$\lim_{k\to\infty} (\mathrm{R}) \int_{[0,k]} |f(x)|\, dx = \lim_{k\to\infty} (\mathrm{L}) \int_{[0,k]} |f(x)|\, dx = (\mathrm{L}) \int_{[0,\infty)} |f(x)|\, dx < \infty$$

Therefore $f(x) \in L([0,\infty))$. Finally, by the conclusion of Theorem 4.26 (let $E_k = [0,k], E = [0,\infty)$), we have

$$\begin{aligned}(\mathrm{L}) \int_{[0,\infty)} f(x)dx &= \lim_{k\to\infty} (\mathrm{L}) \int_{[0,k]} f(x)dx \\ &= \lim_{k\to\infty} (\mathrm{R}) \int_{[0,k]} f(x)dx.\end{aligned}$$

90 **(Example 1)** Give an example of $f(x)$ defined on $(0,\infty)$ which is Riemann improper integralable but is not Lebesgue integrable.

91 **(Example 3)** Find

$$I = \int_0^1 \frac{\ln(x)}{1-x} dx.$$

92 **(Notice)**

(1) Let f be Riemann integrable on $[a,b]$ and let $g(x)$ be bounded on $[a,b]$. Moreover $f(x) = g(x)$ a.e $x \in [a,b]$. Prove or disprove $g(x)$ is Riemann integrable on $[a,b]$.

(2) Let $f(x) \in L([0,1])$ and suppose that $f(x)$ is bounded. Prove or disprove there exists $g(x)$: Riemann integrable on $[0,1]$ s.t $f(x) = g(x)$ a.e $x \in [0,1]$.

(3) Show that there exists $E \subset [a,b]; m(E) = 0$ s.t $\forall f(x) \in R([a,b])$ (a Riemann integrable function on $[a,b]$), E contains at least one point of continuity of f.

93 **(Exercise 4)** Let $f(x) = \sin(x^2)$. Show that f is not Lebesgue integrable on $[0,\infty)$. Hint.

$$\int_{\sqrt{(n-1)\pi}}^{\sqrt{n\pi}} |f(x)|\, dx = \frac{1}{2} \int_{(n-1)\pi}^{n\pi} \frac{|\sin(t)|}{\sqrt{t}} dt \geqq \frac{1}{\sqrt{n\pi}}$$

§ 4.5 Double Integral and Iterated Integral

(I) Fubini's Theorem

Let $\mathscr{F}$ be a family of non-negative Lebesgue measurable functions on $\mathbb{R}^p \times \mathbb{R}^q (= \mathbb{R}^d)$ which satisfy the following conditions.

- (a) $y \mapsto f(x,y)$ is a non-negtive measurable function on $\mathbb{R}^q$ for a.e $x \in \mathbb{R}^p$.
- (b) $F(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^q} f(x,y)dy$ is a non-negative measurable function on $\mathbb{R}^p$.
- (c) $\int_{\mathbb{R}^p} F(x)dx = \int_{\mathbb{R}^d} f(x,y)dxdy$.

94 **(Lemma 4.28)** Before we prove the Tonelli's theore, we prove the following lemma.

(1) Let $f(x,y) \in \mathscr{F}$ and $a \geqq 0$. Show that $a \cdot f(x,y) \in \mathscr{F}$.

(2) Let $f_1(x,y), f_2(x,y) \in \mathscr{F}$. Show that $f_1(x,y) + f_2(x,y) \in \mathscr{F}$.

(3) Let $f_1(x,y), f_2(x,y) \in \mathscr{F}$. Suppose that $f(x,y) - g(x,y) \geqq 0$ and $g(x,y) \in L(\mathbb{R}^d)$. Show that $f(x,y) - g(x,y) \in \mathscr{F}$.

(4) Let $f_k(x,y) \in \mathscr{F}$ and suppose that $f_k(x,y) \nearrow f(x,y)$ as $k \to \infty$. Show that $f(x,y) \in \mathscr{F}$.

(5) Let $f_k(x,y) \in \mathscr{F}$ and suppose that $f_1(x,y) \in L(\mathbb{R}^d)$ and $f_k(x,y) \searrow f(x,y)$ as $k \to \infty$. Show that $f(x,y) \in \mathscr{F}$.

95 **(Theorem 4.27 Tonell's theorem)** Let $f(x,y)$ be a non-negative Lebesgue measurable function defined on $\mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^n$. Show that $f(x,y) \in \mathscr{F}$.

96 **(Theorem 4.28 Fubini's theorem)** Let $f(x,y) \in L(\mathbb{R}^n)$. ($f(x)$ is not necessarily a non-negative measurable function.) Show the following properties.

- $(a*)$ $y \mapsto f(x,y)$ is a measurable function on $\mathbb{R}^q$ for a.e $x \in \mathbb{R}^p$.
- $(b*)$ $F(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^q} f(x,y)dy$ is a measurable function on $\mathbb{R}^p$.
- $(c*)$ $\int_{\mathbb{R}^p} F(x)dx = \int_{\mathbb{R}^d} f(x,y)dxdy$.

97 **(Example 1)** Let $f \in L([0,\infty))$ and let $a > 0$. Show that

$$\lim_{\alpha \to +0, \beta \to \infty} \int_\alpha^\beta \left(\int_0^\infty \sin ax \cdot f(y) \cdot e^{-xy} dy \right) dx = a \int_0^\infty \frac{f(y)}{a^2 + y^2} dy.$$

It is enough for you to prove that we can swap the order of the iterated integrals.

98 **(Example 2)** Show that

$$\int_0^\infty \exp(-x^2)dx = \frac{\sqrt{\pi}}{2}$$

99 **(Exercise 1)** Let $f(x,y) \in L([0,1] \times [0,1])$. Show that

$$\int_0^1 \left(\int_0^x f(x,y)dy \right) dx = \int_0^1 \left(\int_y^1 f(x,y)dx \right) dy.$$

100 **(Exercise 2)** Let $A, B \in \mathscr{M}$. Show that

$$\int_{\mathbb{R}^n} m\left(A_{-\{x\}} \cap B\right) dx = m(A)m(B).$$

(II) Characterization of Lebesgue Integral from a Geometric Viewpoint

101 **(Theorem 4.30)** Let $E \subset \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$. For each $x \in \mathbb{R}^p$, we define $E_{|x} \stackrel{\text{def}}{=} \{y \in \mathbb{R}^q \mid (x, y) \in E\}$.

(1) Let $E \in \mathscr{M}_n$. Show that $E_{|x} \in \mathscr{M}_q$ for a.e $x \in \mathbb{R}^p$. Notice. $\mathscr{M}_n$: a family of Lebesgue measurable sets on $\mathbb{R}^n$.

(2) Show that

$$m_{(n)}(E) = \int_{\mathbb{R}^p} m_{(q)}(E_{|x})dx.$$

102 **(Theorem 4.31)** Let $E_1 \in \mathscr{M}_p$ and let $E_2 \in \mathscr{M}_q$. ($\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$)

(1) Show that $E_1 \times E_2 \in \mathscr{M}_n$.

(2) Show that $m_{(n)}(E_1 \times E_2) = m_{(p)}(E_1)m_{(q)}(E_2)$

103 **(Corollary 4.32)** Let $f(x)$ be a non-negative and real-valued Lebesuge measurable function on $\mathbb{R}^n$ and let $E \in \mathscr{M}; E \subset \mathbb{R}^n$. We define $G(E; f)$ as below. Show that $m_{(n+1)}\left(G(E; f)\right) = 0$.

$$G(E; f) \stackrel{\text{def}}{=} \left\{(x, y) \in \mathbb{R}^{n+1} \mid x \in E, y = f(x)\right\}.$$

104 **(Theorem 4.33 - 1)** Let $f(x) : E \mapsto [0, \infty)$ be a non-negative and real-valued Lebesgue measurable function on $E \in \mathscr{M}; E \subset \mathbb{R}^n$ Let

$$\underline{G}(E; f) \stackrel{\text{def}}{=} \left\{(x, y) \in \mathbb{R}^{n+1} \mid x \in E, 0 \leqq y \leqq f(x)\right\}.$$

Show that $\underline{G}(E; f) \in \mathscr{M}_{(n+1)}$ and

$$m_{(n+1)}\left(\underline{G}(E; f)\right) = \int_E f(x)dx.$$

105 **(Theorem 4.33 - 2)** Let $f(x) : E \mapsto [0, \infty)$ be a non-negative and real-valued function on $E \in \mathscr{M}; E \subset \mathbb{R}^n$. Suppose $\underline{G}(E; f)$ is Lebesgue measurable on $\mathbb{R}^{n+1}$. Show that f is Lebesgue measurable on E.

(III) Convolution and Distribution Function

106 **(Definition of Convolution)** Let $f(x), g(x)$ be Lebesgue measurable functions on $\mathbb{R}^n$. State the definition of $f * g$: convolution of f and g.

107 **(Theorem 4.34)** Let $f, g \in L(\mathbb{R}^n)$.

(1) Show that $(f * g)(x)$ is defined and finite a.e $x \in \mathbb{R}^n$

(2) Show that $(f * g)(x)$ is a Lebesuge measurable function.

(3) Show that

$$\int_{\mathbb{R}^d} |(f * g)(x)|\,dx \leqq \int_{\mathbb{R}^d} |f(x)|\,dx \int_{\mathbb{R}^d} |g(x)|\,dx$$

108 **(Example 5)** Show that there never exists $u(x) \in L(\mathbb{R})$ s.t $\forall f \in L(\mathbb{R})$

$$(u * f)(x) = f(x), \text{a.e } x \in \mathbb{R}$$

109 **(Definition 4.4)** Let $f(x)$ be measurable on $E \in \mathscr{M}$. State the definition of the distribution function of f.

110 **(Theorem 4.35)** Let $f_*(\lambda), \lambda > 0$ be the distribution function of f. Show that $\forall p \in [1, \infty)$,

$$\int_E |f(x)|^p\,dx = p\int_0^\infty \lambda^{p-1} f_*(\lambda) d\lambda.$$

§ 4.6 Exercise

111 **(Exercise 1)** Let $f(x)$ be a measurable function on $E \in \mathscr{M}$. Suppose $f > 0$ a.e $x \in E$ and $\int_E f(x)dx = 0$. Show that $m(E) = 0$.

112 **(Exercise 2)** Let $f(x)$ be non-negative and integrable on $[0, \infty)$ and suppose $f(0) = 0$ and $f'(0)$ exists. Show that the following integral is finite.

$$\int_{[0,\infty)} \frac{f(x)}{x} dx.$$

113 **(Exercise 3)** Let $f(x)$ be non-negative and measurable function on $E \in \mathscr{M}; E \subset \mathbb{R}^n$. There exists a sequence of point sets $\{E_k\}_{k\geq 1}, E_k \subset E; m(E \setminus E_k) < 1/k$ s.t the following limit converges.

$$\lim_{k\to\infty} \int_{E_k} f(x)dx$$

Show that $f(x) \in L(E)$.

114 **(Exercise 4)** Let $f(x)$ be non-negative and integrable on $\mathbb{R}$. We define

$$F(x) \stackrel{\text{def}}{=} \int_{(-\infty,x]} f(t)dt, x \in \mathbb{R}$$

Suppose that $F(x) \in L(\mathbb{R})$. Show that $\int_{\mathbb{R}} f(x)dx = 0$.

115 **(Exercise 5)** Let $f_k(x)$ be a sequence of non-negative and integrable functions on $\mathbb{R}^n$. Suppose $\forall E \in \mathscr{M}$, we have

$$\int_E f_k(x)dx \leqq \int_E f_{k+1}(x)dx.$$

Show that

$$\lim_{k\to} \int_E f_k(x)dx = \int_E \lim_{k\to\infty} f_k(x)dx$$

116 **(Exercise 6)** Let $f(x), g(x)$ be non-negative Lebesgue measurable functions on $E \in \mathscr{M}; E \subset \mathbb{R}; m(E) = 1$. Suppose that $f(x)g(x) \geqq 1$ for all $x \in E$. Show that

$$\int_E f(x)dx \int_E g(x)dx \geqq 1$$

117 **(Exercise 7)** Let $f(x)$ be a function defined on $\mathbb{R}^n$. Suppose that $\forall \epsilon > 0$, $\exists g, h \in L(\mathbb{R}^n)$, s.t $g(x) \leqq f(x) \leqq h(x),\ x \in \mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} (h(x) - g(x))dx < \epsilon.$$

Show that $f \in L(\mathbb{R}^n)$.

118 **(Exercise 8)** Let $\{E_k\}_{k \geqq 1}$ be a sequence of Lebesgue measurable sets with finite measure. Suppose that

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} |\chi_{E_k}(x) - f(x)|dx = 0.$$

Show that there exists a Lebesgue measurable set $E \in \mathscr{M}$ s.t $f(x) = \chi_E(x)$ a.e $x \in \mathbb{R}^n$.

119 **(Exercise 9)** Let $f(x)$ be a bounded monotone increasing function on $[0, 1]$. Show that $\forall E \subset [0, 1]; E \in \mathscr{M}; m(E) = t$,

$$\int_{[0,t]} f(x)dx \leqq \int_E f(x)dx.$$

120 **(Exercise 10)** Let $f \in L(\mathbb{R}^n)$ and let E : be a compact set on $\mathbb{R}^n$. Show that

$$\lim_{|y| \to \infty} \int_{E_{+\{y\}}} |f(x)|dx = 0.$$

Notice. $E_{+\{y\}} \stackrel{\text{def}}{=} \{x + y | x \in E\}$

121 **(Exercise 11)** Show the following equalities.

(1)

$$\frac{1}{\Gamma(\alpha)} \int_{(0,\infty)} \frac{x^{\alpha-1}}{\exp(x) - 1} dx = \sum_{n=1}^{\infty} n^{-\alpha}.$$

(2)

$$\int_{(0,\infty)} \frac{\sin ax}{\exp(x) - 1} dx = \sum_{n=1}^{\infty} \frac{a}{n^2 + a^2}$$

122 **(Exercise 12)** Let $f(x) \in L(\mathbb{R}^1)$ and let $a > 0$. Let

$$S(x) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} f\left(\frac{x}{a} + n\right).$$

(1) Show that $S(x)$ absolutely converges a.e $x \in \mathbb{R}^1$.

(2) $S(x)$ is periodic with period a.

(3) $S \in L([0,a])$

123 **(Exercise 13)** Let $f \in L(\mathbb{R})$ and let $p > 0$. Show that

$$\lim_{n\to\infty} n^{-p} f(nx) = 0,\ \text{a.e } x \in \mathbb{R}.$$

124 **(Exercise 14)** Suppose that $x^s f(x), x^t f(x), s < t$ be integrable on $(0,\infty)$. Show that

$$\int_{[0,\infty)} x^u f(x)dx,\ u \in (s,t)$$

exists and is a continuous function with respect to u.

125 **(Exercise 15)** Let $f(x)$ be a positive valued Lebesgue measurable function on $(0,1)$. Suppose that $\exists c$ s.t

$$\int_{[0,1]} (f(x))^n dx = c,\ (n = 1, 2\cdots).$$

First, show that there exists a Lebesgue measurable set $E \subset (0,1)$ s.t $f(x) = \chi_E(x)$ a.e $x \in (0,1)$. Second, does the same argument hold for $f(x)$ which is not non-negative?

126 **(Exercise 16)** Let $f(x) \in L([0,1])$. Show that

$$\lim_{n\to\infty} \int_{[0,1]} n \ln\left(1 + \frac{|f(x)|^2}{n^2}\right) = 0.$$

Hint. $\ln(1+x^2) \leqq x, x \geqq 0$.

127 **(Exercise 17)** Let $E_1 \supset E_2 \supset \cdots \supset E_k \supset$, let $E \overset{\text{def}}{=} \cap_{k=1}^{\infty} E_k$ and let $f \in L(E_k)$. Show that

$$\lim_{k\to\infty} \int_{E_k} f(x)dx = \int_E f(x)dx.$$

128 **(Exercise 18)** Let $f \in L(E)$ and suppose that $f(x) > 0$ for all $x \in E$. Show that

$$\lim_{k\to\infty} \int_E (f(x))^{\frac{1}{k}} dx = m(E)$$

129 **(Exercise 19)** Let $\{f_n\}_{n\geqq 1} \subset L([0,1])$ be a sequence of non-negative and integrable functions on $[0,1]$. Suppose that $f_n \xrightarrow{m} f(x)$ and

$$\lim_{n\to\infty} \int_{[0,1]} f_n(x)dx = \int_{[0,1]} f(x)dx.$$

Show that $\forall E \in \mathscr{M}, E \subset [0,1]$,

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

130 **(Exercise 20)** Let $\{f_k\}_{k\geqq 1} \subset L(E)$ be a sequene of non-negative and integrable functions on $E \in \mathscr{M}$. Suppose that $f_k(x) \overset{\text{a.e}}{\longrightarrow} f(x) \overset{\text{def}}{=} 0$ and

$$\int_E \max\{f_1(x), \cdots, f_k(x)\}\, dx \leqq M < \infty.$$

Show that

$$\lim_{k\to\infty} \int_E f_k(x)dx = 0.$$

131 **(Exercise 21 Fatou's lemma with convergence in measure)** Let $f_k(x)$ be a sequence of non-negative measurable functions defined on $E \in \mathscr{M}$ and suppose that $f_k \overset{m}{\longrightarrow} f$. Show that

$$\int_E f(x)dx \leqq \liminf_{k\to\infty} \int_E f_k(x)dx.$$

132 **(Exercise 22)** Show that

$$\int_{[0,\infty)} e^{-x^2} \cos 2xt dx = \frac{\sqrt{\pi}}{2} e^{-t^2}, \forall t \in \mathbb{R}.$$

133 **(Exercise 23)** Let $f \in L(\mathbb{R}^n)$ and let $\{f_k\}_{k\geqq 1} \subset L(\mathbb{R}^n)$. Suppose that $\forall E \in \mathscr{M}; E \subset \mathbb{R}^n$, we have

$$\int_E f_k(x)dx \leqq \int_E f_{k+1}(x)dx,\ (k = 1, 2\cdots)$$

and

$$\lim_{k\to\infty} \int_E f_k(x)dx = \int_E f(x)dx.$$

Show that

$$\lim_{k\to\infty} f_k(x) = f(x),\ \text{a.e } x \in \mathbb{R}^n.$$

134 **(Exercise 24)** Let $\{f_k\} \cup \{g_k\}$ be two sequences of measurable functions defined on $E \subset \mathbb{R}; E \in \mathscr{M}$. Suppose $|f_k(x)| \leqq g_k(x)$ for all $x \in E$, $\lim_{k\to\infty} f_k(x) = f(x)$, $\lim_{k\to\infty} g_k(x) = g(x)$ and $\lim_{k\to\infty} \int_E g_k(x)dx = \int_E g(x)dx < \infty$. Show that

$$\lim_{k\to\infty} \int_E f_k(x)dx = \int_E f(x)dx.$$

135 **(Exercise 25)** Let $f(x)$ be a bounded function on $[a, b]$. Let $D \overset{\text{def}}{=} \{x \in [a, b] \mid f \text{ is discontinuous at } x\}$. Suppose D' (limit points of D) is countable. Show that $f(x)$ is Riemann integrable on $[a, b]$.

136 **(Exercise 26)** Let $f(x)$ be a bounded function on $[a, b]$. Suppose that $\forall x \in \mathbb{R}$, $\lim_{h\to 0} f(x+h)$ exists. Show that $f(x)$ is Riemann integrable on any interval $[a, b]$.

137 **(Exercise 27)** Let $E \subset [0, 1]$. Show that $\chi_E(x)$ is Riemann integrable on $[0, 1]$ if and only if $m(\overline{E} \setminus \mathring{E}) = 0$.

138 **(Exercise 28)** Let f be Riemann integrable on $[0,1]$. Show that $f(x^2)$ is also Riemann integrable on $[0,1]$.

139 **(Exercise 29)** Let $f(x), g(x)$ be Lebesgue measurable on $E \subset \mathbb{R}; E \in \mathscr{M}$ and suppose that $m(E) < \infty$. Suppose $f(x) + g(y)$ is integrable on $E \times E$. Show that $f(x), g(x)$ are integrable on E.

140 **(Exercise 30)** Find the following integrals.

(1)

$$\int_{x>0} \int_{y>0} \frac{dxdy}{(1+y^2)(1+x^2y)}.$$

(2)

$$\int_0^\infty \frac{\ln x)}{x^2-1} dx$$

141 **(Exercise 31)** Let $E \subset \mathbb{R}; E \in \mathscr{M}; m(E) > 0$ and let $f(x)$ be a non-negative measurable function on $\mathbb{R}$. Let

$$F(x) \stackrel{\text{def}}{=} \int_E f(x-t)dt.$$

Suppose that $F(x)$ is integrable on $\mathbb{R}$. Show that $f \in L(\mathbb{R})$.

142 **(Exercise 32)** Let $f(x) \in L(\mathbb{R})$ and suppose $xf(x) \in L(\mathbb{R})$. We define

$$F(x) \stackrel{\text{def}}{=} \int_{-\infty}^x f(t)dt$$

. Show that if $\int_{-\infty}^\infty f(x)dx = 0$ then $F \in L(\mathbb{R})$.

143 **(Exercise 33)** Find

$$\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \cos x \arctan(nx)dx$$

144 **(Exercise 34)** Let $I \stackrel{\text{def}}{=} (0,a)$, let $f \in L(I)$ and let $g(x) \stackrel{\text{def}}{=} \int_x^a \frac{f(t)}{t}dt$, $(0 < x < a)$. Show that $g \in L(I)$ and

$$\int_I g(x)dx = \int_I f(x)dx.$$

CHAPTER 5

Differentiation

§ 5.1 Differentiability of Monotone Functions

(I) Vitali's Covering Theorem

1 **(Definition 5.1)** Let $E \subset \mathbb{R}$. Let $\Gamma \overset{\text{def}}{=} \{I_a\}$ be a family of intervals (open, half-open, closed intervals). What does it mean if we say that Γ is a Vitali cover of E?

2 **(Example 1)** Give an example of a Vitali cover of $E \overset{\text{def}}{=} [a, b]$.

3 **(Theorem 5.1 Vitali's Covering Theorem)** Let $E \subset \mathbb{R}$ with $m^*(E) < \infty$. E is not necessarily a Lebesgue measurable set. Suppose that Γ is a Vitali cover of E. Show that there exists a finite number of disjoint $I_1, I_2 \cdots, I_n \in \Gamma$ s.t.

$$m^* \left(E \setminus \bigcup_{j=1}^{n} I_j \right) < \epsilon.$$

(II) Differentiability of Monotone Functions

4 **(Definition 5.2)** Let $f(x)$ be a real-valued function defined on $\mathbb{R}$. State the definition of Dini derivatives $(D^+ f(x_0), D_+ f(x_0), D^- f(x_0), D_- f(x_0))$ at $x = x_0$. State the definition of differentiability based on Dini derivatives.

5 **(Theorem 5.2 Lebesgue's Theorem)** Let $f(x)$ be a real-valued monotone increasing function defined on $[a, b]$.

(1) Show that $f(x)$ is differentiable a.e $x \in [a, b]$. (Show that the set of non-differentiable points of $f(x)$ on $[a, b]$ is a Lebesgue measure zero set.)

(2) Show that

$$\int_{[a,b]} f'(x)dx \leqq f(b) - f(a).$$

6 **(Theorem 5.3 Fubini's Termwise Differentiation Theorem)** Let $\{f_n(x)\}$ be a sequence of monotone-increasing functions on $[a, b]$. Suppose that $\sum_{n=1}^{\infty} f_n(x)$ converges (exists and is finite) on $[a, b]$. Show that

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} f_n(x)\right) = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x) \text{ a.e } x \in [a, b].$$

7 **(Exercise 1)** Let $f(x)$ be a non-negative function defined on $[a, b]$. Suppose that $f \notin L([a, b])$. Does $f(x)$ have a real-valued primitive function? (i.e exists $F(x)$ s.t $F'(x) = f(x)$.)

8 **(Exercise 2)** Let $\{f_n(x)\}_{n \geqq 1}$ be a sequence of monotone increasing functions defined on $(0, 1)$. Suppose that $\lim_{n\to\infty} f_n(x) = 1$ a.e $x \in (0, 1)$. Show that

$$\liminf_{n\to\infty} f_n'(x) = 0 \text{ a.e } x \in (0, 1).$$

9 **(Exercise 3)** Show that we can modify the conclusion of the Vitali's Covering Theorem in the following way. Suppose that Γ is a Vitali cover of $E \subset \mathbb{R}$ with $m^*(E) < \infty$. (E is not necessarily a Lebesgue measurable set.) There exist a countable number of disjoint intervals $\{I_j\}_{j=1}^{\infty} \subset \Gamma$ s.t

$$m^*\left(E \setminus \bigcup_{j=1}^{\infty} I_j\right) = 0.$$

10 **(Exercise 4)** Let $f(x) \in C([a, b])$ be a continuous function defined on $[a, b]$. Show that there exists $x_0 \in (a, b)$ and a constant $k \in \mathbb{R}$ s.t

$$D_- f(x_0) \geqq k \geqq D^+ f(x_0) \text{ or } D^- f(x_0) \leqq k \leqq D_+ f(x_0).$$

11 **(Exercise 5)** Let $E \subset (a, b)$ and suppose that $m(E) = 0$. Construct a continuous and monotone-increasing function $f(x)$ which is defined on $[a, b]$ with $f'(x) = \infty$ for all $x \in E$.

12 **(Exercise 6)** Construct a strictly monotone increasing function $f(x)$ with $f'(x) = 0$ a.e $x \in [0, 1]$.

13 **(Exercise 7)** Let $E \subset \mathbb{R}$. Let I_δ be an open interval whose length is $\delta > 0$ with $x_0 \in I_\delta$. If

$$\lim_{h\to+0} \frac{m^*((x-h, x+h) \cap E^c)}{2h} = 0,$$

then we say that x_0 is a density point of E. Show that if almost every point in E is a density point, then E is Lebesgue measurable. Hint. We may suppose that every point in E is a density point because a measure zero set is measurable. We may also suppose that $E \subset (a, b)$ because if $E_n \overset{\text{def}}{=} E \cap (-n, n) \in \mathscr{M}$ then $E = \bigcup_{n=1}^{\infty} E_n \in \mathscr{M}$.

§ 5.2 Bounded Variation Function

14 **(Definition 5.3)** Let $f(x)$ be a real-valued function defined on $[a,b]$. Let us consider a partition $\Delta \stackrel{\text{def}}{=} \{a = x_0, x_1, \cdots, x_n = b\}$. Let

$$v_\Delta \stackrel{\text{def}}{=} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

Explain what is total variation and what is a bounded variation function defined on $[a,b]$. We denote variation of $f(x)$ on $[a,b]$ as

$$\bigvee_a^b (f).$$

And we also denote the collection of all bounded variation functions defined on $[a,b]$ as $\mathrm{BV}([a,b])$.

15 **(Example 1)** Let $f(x)$ be a monotone function (monotone increasing or monotone decreasing). Find v_Δ.

16 **(Example 2)** Let $f(x)$ be a differentiable function defined on $[a,b]$. Suppose that $|f'(x)| \leqq M < \infty$ for all $x \in [a,b]$. Show that $f(x)$ is a bounded variation function.

17 **(Example 3)** Let

$$f(x) \stackrel{\text{def}}{=} \begin{cases} x \sin \frac{\pi}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}.$$

Show that $f(x)$ is not a bounded variation function defined on $[0,1]$.

18 **(Theorem 5.4)** Let $f(x)$ be a real-valued function defined on $[a,b]$ and let $c \in (a,b)$. Show that

$$\bigvee_a^b (f) = \bigvee_a^c (f) + \bigvee_c^b (f).$$

19 **(Theorem 5.5 Jordan's Decomposition Theorem)** Let $f(x)$ be a real-valued function defined on $[a,b]$. Let $f(x) \in \mathrm{BV}([a,b])$ if and only if $f(x) = g(x) - h(x)$ where $g(x), h(x)$ are real-valued monotone increasing functions on $[a,b]$

20 **(Example 4)** Let $f(x)$ be a real-valued function defined on $[a,b]$. Suppose that $f(x) \in \mathrm{BV}([a,b])$. Show that $f(x)$ is differentiable a.e $x \in [a,b]$ and that

$$\frac{d}{dx}\left(\bigvee_a^b (f)\right) = |f'(x)| \text{ a.e } x \in [a,b].$$

21 **(Example 5)** Let $f(x)$ be a real-valued function defined on $[a,b]$. Suppose that $f(x) \in \mathrm{BV}([a,b])$. Let ℓ_f be a length of the curve $y = f(x)$ $(x \in [a,b])$. Show that

$$\ell_f \geqq \int_a^b \sqrt{1 + \{f'(x)\}^2}.$$

22 **(Exercise 1)** Find

$$\bigvee_{-1}^{1}(x - x^3).$$

23 **(Exercise 2)** Show that

$$\bigvee_{a}^{b}(f) = 0$$

if and only if $f(x) = C$ where C is a constant.

24 **(Exercise 3)** Let $f(x), g(x) \in \mathrm{BV}([a, b])$. Show that $M(x) \stackrel{\text{def}}{=} \{f(x), g(x)\}$ is a bounded variation function defined on $[a, b]$.

25 **(Exercise 4)** Show that $f(x) \in \mathrm{BV}([a, b])$ implies that $|f(x)| \in \mathrm{BV}([a, b])$, however $|f(x)| \in \mathrm{BV}([a, b])$ does not imply that $f(x) \in \mathrm{BV}([a, b])$.

26 **(Exercise 5)** Let $f(x), g(x) \in \mathrm{BV}([a, b])$. Show that

$$\bigvee_{a}^{b}(fg) \leqq \sup_{x\in[a,b]}\{f(x)\} \cdot \bigvee_{a}^{b}(g) + \sup_{x\in[a,b]}\{g(x)\} \cdot \bigvee_{a}^{b}(f)$$

27 **(Exercise 6)** Let $f(x) \in \mathrm{BV}([a, b])$ and let $\phi(x)$ be a Lipschitz continuous function (i.e $|\phi(x_1) - \phi(x_2)| \leqq L|x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}$ for some L.) Show that $\phi \circ f(x) \in \mathrm{BV}([a, b])$.

28 **(Exercise 7)** Let $f(x)$ be a Lipschitz continuous function defined on $[a, b]$. (i.e $|f(x_1) - f(x_2)| \leqq L|x_1 - x_2|$ for all $x_1, x_2 \in [a, b]$ for some L.) Show that

$$g(x) \stackrel{\text{def}}{=} \bigvee_{a}^{x}(f).$$

is also a Lipschitz continuous function defined on $[a, b]$.

29 **(Exercise 8)** Show that $f(x) \in \mathrm{BV}([a, b])$ if and only if there exists a monotone increasing function $F(x)$ define on $[a, b]$ s.t

$$|F(x_1) - F(x_2)| \leqq F(x_2) - F(x_1) \ (a \leqq x_1 < x_2 \leqq b)$$

30 **(Exercise 9)** Let $f(x) \in \mathrm{BV}([a, b])$. Suppose that $f(x)$ has a primitive functgion on $[a, b]$. Discuss if $f(x)$ is continuous on $[a, b]$.

31 **(Exercise 10)** Let $f(x) \in \mathrm{BV}([a, b])$. Suppose that

$$\bigvee_{a}^{b}(f) = f(b) - f(a).$$

Show that $f(x)$ is monotone-increasing on $[a, b]$.

32 **(Exercise 11)** Let $\{f_n(x)\} \subset \mathrm{BV}([a,b])$. Suppose that $\sum_{n=1}^{\infty} f_n(x)$ and $\sum_{n=1}^{\infty} \bigvee_a^x(f_n)$ converges for all $x \in [a,b]$. Show that $f(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} f_n(x)$ is a bounded variation function defined on $[a,b]$.

§ 5.3 Differentiation of Indefinite Integral

Let $f(x) \in L([a,b])$ and let $F(x) \stackrel{\text{def}}{=} \int_a^x f(t)dt$. In this section, we are going to discuss if $\frac{d}{dx}F(x) = f(x)$ holds.

33 **(Lemma 5.6)** Let $f(x) \in L([a,b])$ and let

$$F_h(x) \stackrel{\text{def}}{=} \frac{1}{h}\int_x^{x+h} f(t)dt.$$

Suppose that $f(x) = 0$ if $x \notin [a,b]$. Show that

$$\lim_{h\to 0}\int_a^b |F_h(x) - f(x)|dx = 0.$$

34 **(Theorem 5.7)** Let $f(x) \in L([a,b])$ and let

$$F(x) \stackrel{\text{def}}{=} \int_a^x f(t)dt, \ x \in [a,b].$$

Show that

$$F'(x) = f(x) \text{ a.e } x \in [a,b].$$

35 **(Corollary 5.8)** Let $f(x) \in L([a,b])$. Show that

$$\lim_{h\to 0}\frac{1}{h}\int_0^h |f(x+t) - f(x)|dt = 0 \text{ a.e } x \in [a,b].$$

When the equality above holds, we say that x is a Lebesgue point.

36 **(Example 1)** Let $f(x) \in L(\mathbb{R})$. Suppose that

$$\int_a^b |f(x+h) - f(x)|dx = o(|h|) \text{ as } h \to 0,$$

for $x \in [a,b]$. ($= o(|h|)$ means that the left hand side goes to 0 faster than $|h|$ when $h \to 0$.) Show that

$$f(x) = C \text{ (constant)}$$

37 **(Example 2)** Let $f(x) \in L([a,b])$ and let

$$F(x) \stackrel{\text{def}}{=} \int_a^x f(t)dt \ (x \in [a,b]).$$

(1) Show that

$$F(x) \in \mathrm{BV}([a,b]).$$

(2) Show that

$$\bigvee_a^b (F) \leqq \int_a^b |f(x)|dx.$$

38 **(Exercise 1)** Let $E \subset [0,1]$ be a Lebesgue measurable set. Suppose that there exists $\ell \in (0,1)$ s.t for any closed interval $[a,b] \subset [0,1]$, the following inequality holds,

$$m(E \cap [a,b]) \geqq \ell(b-a).$$

Show that $m(E) = 1$.

39 **(Exercise 2)** Let us consider a Dirichlet function $\chi_{\mathbb{Q}}(x)$ defined on $x \in [0,1]$. Find the Lebesgue points on $[0,1]$.

§ 5.4 Absolutely Continuous Function and Fundamental Theorem of Calculus

In this section, we are going to discuss if the following equality holds,

$$f(x) - f(a) = \int_a^x f'(t)dt \ x \in [a,b].$$

40 **(Lemma 5.9)** Let $f(x)$ be a function defined on $[a,b]$ and suppose that $f(x)$ is differentiable a.e $x \in [a,b]$ and that $f'(x) = 0$ a.e $x \in [a,b]$. Show that if $f(x)$ is not a constant function, then there exists a positive number $\epsilon > 0$ s.t for any positive number $\delta > 0$, we can find a finite number of disjoint open intervals $\{(x_i, y_i)\}_{i=1}^n$ satisfying

- $y_i - x_i < \delta$ for all $i = 1, 2, \cdots, n$,
- $\sum_{i=1}^n |f(y_i) - f(x_i)| > \epsilon$.

41 **(Definition 5.4)** Let $f(x)$ be a real-valued function defined on $[a,b]$. What does it mean if we say that $f(x)$ is an absolutely continuous function on $[a,b]$. We denote the collection of all absolutely continuous functions defined on $[a,b]$ as AC($[a,b]$).

42 **(Example 1)** Let $f(x)$ be a Lipschitz continuous function defined on $[a,b]$. Verify that $f(x)$ is absolutely continuous on $[a,b]$.

43 **(Theorem 5.10)** Let $f(x) \in L([a,b])$. Show that

$$F(x) \stackrel{\text{def}}{=} \int_a^x f(t)dt$$

is an absolutely continuous function defined on $[a,b]$.

44 **(Theorem 5.11)** Let $f(x)$ be an absolutely continuous function defined on $[a,b]$. Show that $f(x)$ is a bounded variation function.

45 **(Corollary 5.12)** Let $f(x)$ be an absolutely continuous function defined on $[a,b]$. Show that $f(x)$ is differentiable a.e $x \in [a,b]$ and also show that $f'(x)$ is an integrable function on $[a,b]$.

46 **(Theorem 5.13)** Let $f(x)$ be an absolutely continuous functions defined on $[a,b]$ and suppose that $f'(x) = 0$ a.e $x \in [a,b]$. Show that $f(x) = C$ (a constant) on $[a,b]$.

47 **(Theorem 5.14 A Fundamental Theorem of Calculus)** Let $f(x)$ be an absolutely continuous function on $[a,b]$. Show that

$$f(x) - f(a) = \int_a^x f'(t)dt, \ x \in [a,b].$$

48 **(Example 2)** Let $g_k(x)$ be an absolutely continuous function on $[a,b]$. We suppose that

- there exists $c \in [a,b]$ s.t $\sum_{k=1}^{\infty} g_k(c)$ converges,
- $\sum_{k=1}^{\infty} \int_a^b |g'_k(x)|dx < \infty$.

(1) Show that $g(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} g_k(x)$ converges on $x \in [a,b]$.

(2) Show that $g(x)$ is absolutely continuous on $[a,b]$ and also show that

$$g'(x) = \sum_{k=1}^{\infty} g'_k(x) \text{ a.e } x \in [a,b].$$

49 **(Example 4)** Let $f(x)$ be absolutely continuous on $[a,b]$. Show that the length of curve is

$$\ell_f = \int_a^b \sqrt{1 + f'(x)^2}dx.$$

50 **(Example 5)** Let $f(x) \in L([c,d])$ where $[a,b] \subset [c,d]$. ($c < a < b < d$.) Suppose that

$$\int_a^b |f(x+h) - f(x)|dx \sim o(|h|), \text{ as } h \to 0.$$

Show that there exists $g(x) \in \mathrm{BV}([a,b])$ s.t $f(x) = g(x)$ a.e $x \in [a,b]$.

51 **(Example 8)** Let $f(x)$ be differentiable on $\mathbb{R}$ and suppose that $|f'(x)| \leqq < \infty$. Suppose that $\{x \in \mathbb{R} \mid f'(x) > 0\}$ and $\{x \in \mathbb{R} \mid f'(x) < 0\}$ are dense on $\mathbb{R}$. Show that $f'(x)$ is not Riemann integrable on $[a,b]$ where $[a,b] \subset \mathbb{R}$ is an arbitrary closed interval.

52 **(Example 9)** Let $f(x)$ be absolutely continuous on $[a,b]$. Show that $m(f(Z)) = 0$ for all $Z \subset [a,b]$ with $m(Z) = 0$.

53 **(Example 10)** Let $f(x) \in C([a,b]) \cap \mathrm{BV}([a,b])$. Suppose that $m(f(Z)) = 0$ for all $Z \subset [a,b]$ with $m(Z) = 0$. Shpow that $f(x)$ is absolutely continuous on $[a,b]$.

54 **(Exercise 1)** Let $f(x)$ be absolutely continuous on $[a, b]$ and suppose that $|f'(x)| \leqq M < \infty$ a.e $x \in [a, b]$. Show that $|f(y) - f(x)| \leqq M|x - y|$ for all $x, y \in [a, b]$.

55 **(Exercise 2)** Let $f(x)$ be a function defined on $[a, b]$. Suppose that $|f(y) - f(x)| \leqq M|y - x|$ for all $x, y \in [a, b]$. Show that $|f'(x)| \leqq M$ a.e $x \in [a, b]$.

56 **(Exercise 3)** Let $\{f_n(x)\}_{n \geqq 1}$ be a sequence of absolutely continuous and monotone increasing functions. Suppose that $\sum_{n=1}^{\infty} f_n(x)$ converges on $[a, b]$. Show that $\sum_{n=1}^{\infty} f_n(x)$ is absolutely continuous on $[a, b]$.

57 **(Exercise 4)** Let $f(x) \in \mathrm{BV}([0, 1])$. Suppose that for all $\epsilon \in (0, 1)$, $f(x)$ is absolutely continuous on $[\epsilon, 1]$, and $f(x)$ is continuous at $x = 0$. Show that $f(x)$ is absolutely continuous on $[0, 1]$.

58 **(Exercise 5)** Show that there exist a strictly monotone increasing absolutely continuous function $f(x)$ and a Lebesgue measurable set $E \in \mathscr{M}, E \subset [0, 1]$ with $m(E) > 0$ s.t $f'(x) = 0$ for all $x \in E$. Hint. Construct a Cantor-Like set C_α with $m(C_\alpha) = 1 - \alpha > 0$ and let $f(x) \overset{\mathrm{def}}{=} \int_0^x \chi_{[0,1] \setminus C_\alpha}(t)dt$.

§ 5.5 Formula of Integral by Parts and Mean Value Theorem of Integral

59 **(Theorem 5.15 Formular of Integral by Parts)** Let $f(x), g(x)$ be integral functions defined on $[a, b]$ and let $\alpha, \beta \in \mathbb{R}$. Let $F(x) \overset{\mathrm{def}}{=} \alpha + \int_a^x f(t)dt$ and let $G(x) \overset{\mathrm{def}}{=} \beta + \int_a^x g(t)dt$. Show that

$$\int_a^b G(x)f(x)dx + \int_a^b g(x)F(x)dx = F(b)G(b) - F(a)G(a).$$

60 **(Theorem 5.16 The First Intermediate Value Theorem in Integral)** Let $f(x) \in C([a, b])$ and let $g(x)$ be a non-negative integrable function defined on $[a, b]$. Show that there exists $\xi \in [a, b]$ s.t

$$\int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx.$$

61 **(Theorem 5.17 The Second Intermediate Value Theorem in Integral)** Let $f(x) \in L([a, b])$ and let $g(x)$ be a monotone increasing (or monotone decreasing) function defined on $[a, b]$. Show that there exists $\xi \in [a, b]$ s.t

$$\int_a^b f(x)g(x)dx = g(a) \int_a^\xi f(x)dx + g(b) \int_\xi^b f(x)dx.$$

62 **(Exercise 1)** Let $f(x) \in L([a, b])$ and let $g(x) = f(x) \int_a^x f(t)dt$. Show that

$$\int_a^b g(x)dx = \frac{1}{2} \left(\int_a^b f(x)dx \right)^2.$$

63 **(Exercise 2)** Let $f(x), g(x)$ be measurable functions defined on $[0,\infty)$. Suppose that $|f(x)| \leqq M < \infty$ for all $x \in [1,\infty)$, and that $|xg(x)| \leqq M < \infty$ for all $x \in [1,\infty)$. Show that

$$\lim_{x\to\infty} \frac{1}{x}\int_1^x f(t)g(t)dt = 0.$$

64 **(Exercise 3)** Let $g(x) \in L(\mathbb{R})$. Let $f(x) \in C^{(2)}(\mathbb{R})$ (twice differentiable and $f''(x)$ is continuous on $\mathbb{R}$) with $f(x) = 0$ for all $x \notin (a,b)$. Show that there exists $C > 0$ s.t

$$\left|\int_{\mathbb{R}} g(x)f(x)^2 dx\right| \leqq C\int_{\mathbb{R}} \left(f(x)^2 + f'(x)^2\right) dx$$

65 **(Exercise 4)** Let $f(x) \in L([a,b])$ and let $F(x) \stackrel{\text{def}}{=} \int_a^x f(t)\cdot(x-t)^n dt$ ($x \in [a,b]$).

(1) $F(x)$ is differentiable n times.

(2) Show that $F^{(n)}$ is absolutely continuous on $[a,b]$.

(3) Show that $F^{(n+1)}(x) = n!f(x)$ a.e $x \in [a,b]$.

§ 5.6 Change of Variable Formula on $\mathbb{R}$

Let $g : [a,b] \mapsto [c,d]$ be differentiable a.e $x \in [a,b]$. We start to discuss if the following change of variable formula holds or not.

$$\int_{g(\alpha)}^{g(\beta)} f(x)dx = \int_\alpha^\beta f(g(t))g'(t)dt, \ [\alpha,\beta] \subset [a,b].$$

66 **(Theorem 5.18)** Let $f(x)$ be an absolutely continuous function defined on $[a,b]$, and let $E \subset [a,b], E \in \mathscr{M}$. Show that $f(E) \stackrel{\text{def}}{=} \{f(x) \mid x \in E\} \in \mathscr{M}$

67 **(Lemma 5.19)** Let $f(x)$ be a real-valued function on $[a,b]$ and let $E \subset [a,b]$. Suppose that $f'(x)$ exists at every $x \in E$ and $|f'(x)| \leqq M < \infty$. Show that

$$m^*(f(E)) \leqq Mm^*(E).$$

68 **(Corollary 5.20)** Let $f(x)$ be a measurable function on $[a,b]$ and let $E \subset [a,b], E \in \mathscr{M}$. Suppose that $f(x)$ is differentiable on E. Show that

$$m^*(f(E)) \leqq \int_E |f'(x)|dx.$$

69 **(Example 1)** Let $f(x)$ be differentiable a.e $x \in [a,b]$ and suppose that $f'(x)$ is integrable on $[a,b]$. Show that

$$\int_a^b f'(x)dx = f(b) - f(a).$$

70 **(Theorem 5.21)** Let $f(x)$ be a real-valued function on $[a,b]$ and suppose that $f(x)$ is integrable on $E \in \mathscr{M}, E \subset [a,b]$.

(1) Show that if $f'(x) = 0$ a.e $x \in E$, then $m(f(E)) = 0$.

(2) Show that if $m(f(E)) = 0$, then $f'(x) = 0$ a.e $x \in E$.

71 **(Theorem 5.22 Differentiation of Composite Function)** Let $g : [a,b] \mapsto [c,d]$ be differentiable a.e $x \in [a,b]$. Let $F(x)$ be differentiable a.e $x \in [c,d]$ and suppose that $F'(x) = f(x)$ a.e $x \in [c,d]$. Suppose that $F \circ g(t)$ is differentiable a.e $x \in [a,b]$. Suppose that $m(F(Z)) = 0$ for all $Z \subset [c,d]$ with $m(Z) = 0$. Show that

$$(F(g(t)))' = f(g(t))g'(t) \text{ a.e } t \in [a,b]$$

72 **(Corollary 5.23)** Let $g(t)$, $f \circ g(t)$ be differentiable a.e $x \in [a,b]$ where $f(x)$ is absolutely continuous on $[c,d]$ and suppose that $g([a,b]) \subset [c,d]$. Show that

$$(f(g(t)))' = f'(g(t))g'(t) \text{ a.e } x \in [a,b]$$

73 **(Theorem 5.24 Change of Variable Formula)** Let $g(x)$ be differentiable a.e $x \in [a,b]$ and let $f(x)$ be an integrable function on $[c,d]$. Suppose that $g([a,b]) \subset [c,d]$. Let $F(x) \overset{\text{def}}{=} \int_c^x f(t)dt$. Show that the following statements are equivalent.

- $F(g(t))$ is absolutely continuous on $[a,b]$.
- $f(g(t)) \cdot g'(t)$ is integrable on $[a,b]$ and $\int_{g(\alpha)}^{g(\beta)} f(x)dx = \int_\alpha^\beta f(g(t)) \cdot g'(t)dt$.

74 **(Corollary 5.25)** Let $g(x) : [a,b] \mapsto [c,d]$ be an absolutely continuous function and let $f(x) \in L([c,d])$. Show that each following statement is a sufficient condition for

$$\int_{g(\alpha)}^{g(\beta)} f(x)dx = \int_\alpha^\beta f(g(t)) \cdot g'(t)dt.$$

(1) $g(t)$ is monotone increasing (or decreasing) on $[a,b]$

(2) $f(x)$ is bounded on $[c,d]$.

(3) $f \circ g(t) \cdot g'(t)$ is integrable on $[a,b]$.

75 **(Example 2)** Let $f(x)$ be a non-negative monotone decreasing function defined on $[0,\infty)$. Suppose that for all $A > 0$, $f(x)$ is absolutely continuous on $[0,A]$. Show that

$$p\int_0^\infty (f(x))^p \cdot x^{p-1}dx \leqq \left(\int_0^\infty f(x)dx\right)^p, \ (p \geqq 1)$$

§ 5.7 Exercises

76 **(Exercise 1)** Let $E \subset \mathbb{R}$ be a union of intervals (open, closed or half-open). Show that E is measurable.

77 **(Exercise 2)** Let $\{x_n\} \subset [a,b]$. Construct a monotone increasing function whose points of discontinuity are $\{x_n\}$.

78 **(Exercise 3)** Let $f(x)$ be a monotone increasing function and let $E \subset (a,b)$. Suppose that $\forall\epsilon > 0$, there exists $\{(a_i,b_i)\}_{i\in\mathbb{N}}$ with $(a_i,b_i) \subset (a,b)$ s.t

$$E \subset \bigcup_{i=1}^{\infty}(a_i,b_i),\ \sum_{i=1}^{\infty}(f(b_i)-f(a_i)) < \epsilon.$$

Show that

$$f'(x) = 0 \text{ a.e } x \in E.$$

79 **(Exercise 4)** $f(x)$ is a bounded variation function on $[0,\alpha]$. Show that

$$F(x) \overset{\text{def}}{=} \frac{1}{x}\int_0^x f(y)dy,\ F(0) \overset{\text{def}}{=} 0,$$

is a bounded variation function on $[0,\alpha]$.

80 **(Exercise 5)** Let $\{f_k(x)\}_{k\in\mathbb{N}}$ is a sequence of bounded variation functions. Suppose that

$$\bigvee_a^b(f_k) \leqq M \text{ for each } k \in \mathbb{N},$$

and

$$\lim_{k\to\infty} f_k(x) = f(x),\ x \in [a,b].$$

Show that $f(x)$ is a bounded variation function on $[a,b]$ and also

$$\bigvee_a^b(f) \leqq M$$

81 **(Exercise 6)** Let $f(x)$ be a bounded variation function defined on $[a,b]$, and suppose that $x_0 \in [a,b]$ is a point of continuity of $f(x)$. Show that $g(x) \overset{\text{def}}{=} \bigvee_a^x(f)$ is continuous at $x = x_0$.

82 **(Exercise 7)** Let $f : [a,b] \mapsto [c,d]$ be a continuous function and suppose that for every $y \in [c,d]$, $f^{-1}(\{y\})$ contains at least 10 points. Show that

$$\bigvee_a^b(f) \leqq 10(d-c).$$

83 **(Exercise 8)** Let $f(x) \in L([0,1])$ and let $g(x)$ be a monotone increasing function defined on $[0,1]$. Suppose that for every $[a,b] \subset [0,1]$,

$$\left|\int_a^b f(x)dx\right|^2 \leqq (g(b)-g(a))\cdot(b-a).$$

Show that $f(x)^2$ is an integrable function on $[0,1]$.

84 **(Exercise 9)** Let $f(x)$ be a non-negative absolutely continuous function on $[a,b]$. Show that $f(x)^p$ $(p > 1)$ is an absolutely continuous function on $[a,b]$.

85 **(Exercise 10)** Let $f(x)$ be a monotone increasing function on $[a,b]$, and suppose that

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Show that $f(x)$ is absolutely continuous on $[a,b]$.

86 **(Exercise 11)** Let $f(x) \in \mathrm{BV}([a,b])$. Suppose that

$$\int_a^b |f'(x)|dx = \bigvee_a^b(f).$$

Show that $f(x)$ is absolutely continuous on $[a,b]$.

87 **(Exercise 12)** Let $f(x)$ be a monotone increasing and bounded function on $\mathbb{R}$. Suppose that $f(x)$ is differentiable on $\mathbb{R}$. Let $A \stackrel{\text{def}}{=} \lim_{x\to-\infty} f(x)$ and let $B \stackrel{\text{def}}{=} \lim_{x\to+\infty} f(x)$. Show that

$$\int_{\mathbb{R}} f'(x)dx = B - A.$$

88 **(Exercise 13)** Let $f(x)$ be a differentiable function on $\mathbb{R}$ and suppose that both $f(x), f'(x)$ are integrable on $\mathbb{R}$. Show that

$$\int_{\mathbb{R}} f'(x)dx = 0.$$

89 **(Exercise 14)** Let $f(x,y)$ be a function defined on $[a,b]\times[c,d]$. Suppose that there exists $y_0 \in (c,d)$ s.t $f(x,y_0)$ is integrable on $[a,b]$, and suppose that for every fixed $x \in [a,b]$, $f(x,y)$ as a function of y, (i.e $y \mapsto f(x,y)$) is absolutely continuous, and also suppose that $f'_y(x,y) \stackrel{\text{def}}{=} \frac{\partial}{\partial y} f(x,y)$ is integrable on $[a,b]\times[c,d]$.

(1) Show that

$$F(y) = \int_a^b f(x,y)dx$$

is absolutely continuous on $[c,d]$.

(2) Show that

$$F'(y) = \int_a^b f'_y(x,y)dx \text{ a.e } y \in [c,d].$$

90 **(Exercise 15)** Let $f(x)$ be absolutely continuous on every $[a,b] \subset \mathbb{R}$. Show that for every $y \in \mathbb{R}$, we have

$$\frac{\partial}{\partial y}\int_a^b f(x+y)dx = \int_a^b \frac{\partial}{\partial y} f(x+y)dx.$$

91 **(Exercise 16)** Explain that we can no longer improve the proposition that an absolutely continuous function is differentiable almost everywhere by giving an example.

92 **(Exercise 17)** Let $\{g_k(x)\}$ be a sequence of absolutely continuous functions on $[a, b]$ with $|g_k'(x)| \leqq F(x)$ a.e $x \in [a, b]$ where $F(x) \in L([a, b])$. Suppose that $\lim_{k\to\infty} g_k(x) = g(x)$ a.e $x \in [a, b]$, and $\lim_{k\to\infty} g_k'(x) = f(x)$ a.e $x \in [a, b]$. Show that

$$g'(x) = f(x) \text{ a.e } x \in [a, b].$$

93 **(Exercise 18)** Let $f(x)$ be an absolutely continuous and strictly monotone increasing function. Let $g(y)$ be absolutely continuous on $[f(a), f(b)]$. Show that $g \circ f(x)$ is absolutely continuous on $[a, b]$.

94 **(Exercise 19)** Let $g(x)$ be absolutely continuous on $[a, b]$ and suppose that $f(x)$ is Lipschitz continuous on $\mathbb{R}$. Show that $f \circ g(x)$ is absolutely continuous on $[a, b]$.

95 **(Exercise 20)** Suppose that $f(x)$ is differentiable on $[a, b]$. Show that if $f'(x) = 0$ a.e $x \in [a, b]$, then $f(x)$ is a constant function.

CHAPTER 6

L^p space

§ 6.1 Definition of L^p space and some Inequalities

1 **(Definition 6.1)** Let $f(x)$ be a Lebesgue measurable function on $E \subset \mathbb{R}^d, E \in \mathscr{M}$.

(1) Define $\|f\|_p$ ($p \in (0, \infty)$).

(2) Explain what is $L^p(E)$.

(3) Explain what it means if we say that $f(x)$ is essentially bounded on E.

(4) Explain what is essential supremum of $f(x)$ and define $\|f\|_\infty, L^\infty(E)$.

2 **(Property)** Let $f(x)$ be a Lebesgue measurable function on $E \subset \mathbb{R}^d, E \in \mathscr{M}$ and suppose that $m(E) > 0$. Show that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

3 **(Theorem 6.1)** Let $f(x), g(x) \in L^p(E)$ where $p \in (0, \infty]$. Let $\alpha, \beta \in \mathbb{R}$. Show that

$$\alpha f(x) + \beta g(x) \in L^p(E).$$

4 **(Exercise 1)** Suppose that $E \in \mathscr{M}$, $0 < m(E) < \infty$. Let $\{p_k\} \subset (1, \infty)$ with $1 < p_1 < p_2 < \cdots < p_k \to \infty$ as $k \to \infty$. Suppose that $f(x) \in L^{p_k}(E)$ for every $k \in \mathbb{N}$ and $\sup_{k \in \mathbb{N}}\{\|f\|_{p_k}\} < \infty$. Show that $f(x) \in L^\infty(E)$.

5 **(Exercise 2)** Let $0 < p < q$. Show that if $f(x) \in L^p(E) \cap L^\infty(E)$, then $f(x) \in L^q(E)$.

6 **(Exercise 3)** Let $f(x) \in L^1(E) \cap L^2(E)$.

$$\lim_{p \nearrow p_0} \int_E |f(x)|^p dx = \int_E |f(x)|^{p_0} dx.$$

7 **(Exercise 4)** Let $E \in \mathscr{M}, m(E) < \infty$ and let $f(x)$ be a measurable function defined on E. Show that

$$\lim_{p \searrow 1} \int_E |f(x)|^p dx = \int_E |f(x)| dx.$$

8 **(Definition 6.2 Conjugate index)** What is a conjugate index (conjugate indices)?

9 **(Theorem 6.2 Hölder's Inequality)** Let p, q be conjugate indices. Suppose that $f(x) \in L^p(E), g(x) \in L^q(E)$. ($E \in \mathscr{M}$) Show that

$$\|fg\| \leqq \|f\|_p \cdot \|g\|_q$$

10 **(Notice)** Discuss if Hölder's inequality holds if $\|f\|_p = \infty$ or $\|g\|_q = \infty$ holds.

11 **(Example 2)** Suppose that $m(E) < \infty, E \in \mathscr{M}$ and $0 < p_1 < p_2 \leqq \infty$.

(1) Show that $L^{p_2}(E) \subset L^{p_1}(E)$.

(2) Show that

$$\|f\|_{p_1} \leqq (m(E))^{1/p_1 - 1/p_2} \cdot \|f\|_{p_2}.$$

12 **(Example 3)** Let $f(x) \in L^r(E) \cap L^s(E)$ and let $0 < r < p < s \leqq \infty$. Let $\lambda \in (0,1)$ be a number to satisfy $\frac{1}{p} = \frac{\lambda}{r} + \frac{1-\lambda}{s}$. Show that

$$\|f\|_p \leqq \|f\|_r^{\lambda} \cdot \|f\|_s^{1-\lambda}.$$

13 **(Example 4)** Let $0 < r < p < s < \infty$ and let $f(x) \in L^p(E)$. ($E \in \mathscr{M}$). Show that for all $t > 0$, there exists a decomposition $f(x) = g(x) + h(x)$ s.t

$$\|g\|_r^r \leqq t^{r-p} \cdot \|f\|_p^p \text{ and } \|h\|_s^s \leqq t^{s-p} \|f\|_p^p.$$

14 **(Example 5 Inverse Hölder's Inequality)** Let $0 < p < 1$, $q < 0$ and suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f(x) \in L^p(E)$ and $g(x) \in L^q(E)$. ($E \in \mathscr{M}$). Show that

$$\int_E |f(x)g(x)| dx \geqq \|f\|_p \cdot \|g\|_q.$$

15 **(Exercise 5)** Let $f(x), g(x)$ be measurable functions defined on $E \in \mathscr{M}$. Suppose that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. ($1 \leqq p < \infty$). Show that

$$\|fg\|_r \leqq \|f\|_p \cdot \|g\|_q$$

16 **(Exercise 6)** Let $f(x) \in L^2((0,\infty))$ and $f(x) \geqq 0$ for all $x \in (0,\infty)$. Let $F(x) \stackrel{\text{def}}{=} \int_0^x f(t)dt$. Show that

$$F(x) \sim o(\sqrt{x}) \ (x \to +0).$$

($F(x)$ goes to 0 faster than $\sqrt{x}$.)

17 **(Exercise 7)** Let $f(x) \in L^2([0,1])$. Show that there exists a monotone increasing function $g(x)$ s.t for every $[a,b] \subset [0,1]$,

$$\left| \int_a^b f(x)dx \right|^2 \leqq (g(b) - g(a))(b - a).$$

18 **(Exercise 8)** Let $f(x) \in L^2([0,1])$ and suppose that $\|f\|_2 \neq 0$. Let $F(x) \overset{\text{def}}{=} \int_0^x f(t)dt$, $(x \in [0,1])$. Show that

$$\|F\|_2 < \|f\|_2$$

19 **(Theorem 6.3 Minkovski's Inequality)** Let $f(x), g(x) \in L^p(E)$ where $1 \leqq p \leqq \infty$. Show that

$$\|f + g\|_p \leqq \|f\|_p + \|g\|_p.$$

20 **(Example 6 Inverse Mikovski's Inequality)** Let $0 < p < 1$ and let $f(x), g(x) \in L^p(E)$. $(E \in \mathscr{M})$. Show that

$$\||f| + |g|\|_p \geqq \|f\|_p + \|g\|_p.$$

21 **(Notice 1)** Let $f(x) \in L^{p_1}(E), g(x) \in L^{p_2}(E)$ where $0 < p_1 < p_2 < \infty$. Show that

$$f(x)g(x) \in L^p(E) \text{ where } \frac{1}{p} \overset{\text{def}}{=} \frac{1}{p_1} + \frac{1}{p_2}$$

22 **(Notice 2)** Let $f(x) \in L^1(\mathbb{R})$ be a differentiable function and suppose that $f'(x) \in L^p(\mathbb{R})$ where $p > 1$. Show that

$$\lim_{|x| \to \infty} f(x) = 0.$$

23 **(Notice 5)** Let $f(x) \in L^p([0,1])$ where $p > 0$. Show that

$$\lim_{p \to +0} \|f\|_p = \exp\left(\int_0^1 \ln(|f(x)|)dx \right)$$

24 **(Exercise 9)** Let $1 \leqq p \leqq \infty$ and let $\{f_k(x)\}_{k \in \mathbb{N}} \subset L^p(E)$. Suppose that $\sum_{k=1}^{\infty} f_k(x)$ converges a.e $x \in E$. Show that

$$\left\| \sum_{k=1}^{\infty} f_k \right\|_p \leqq \sum_{k=1}^{\infty} \|f_k\|_p.$$

25 **(Exercise 10)** Let $f(x) \in L^p(E)$ where $p \geqq 1$ and $E \in \mathscr{M}$. Let $e \in \mathscr{M}$ with $e \subset E$. Show that

$$\left(\int_E |f(x)|^p \right)^{1/p} \leqq \left(\int_e |f(x)|^p \right)^{1/p} + \left(\int_{E \setminus e} |f(x)|^p \right)^{1/p}.$$

§ 6.2 Structure of L^p space

Let us recall the definition of a metric space. Let $d : X \mapsto [0, \infty)$ and suppose that

- $d(x, y) \geqq 0$ for all $x, y \in X$,
- $d(x, y) = 0$ if and only if $x = y \in X$,
- $d(x, y) = d(y, x)$ for all $x, y \in X$,
- $d(x, y) \leqq d(x, z) + d(z, x)$ for all $x, y, z \in X$.

Then (X, d) is called a metric space.

(I) $L^p(E)$ as a complete metric space

We define

$$L^p(E) \overset{\text{def}}{=} \left\{ f(x) \mid \int_E |f(x)|^p dx < \infty \right\},$$

where $E \in \mathscr{M}$ and $f(x)$ is a Lebesgue measurable function defined on E.

26 **(Theorem 6.4)** Let $f(x), g(x) \in X \overset{\text{def}}{=} L^p(E)$ and let $d(f, g) \overset{\text{def}}{=} \|f - g\|_p$ where $p \in [1, \infty]$. Show that (X, d) is a metric space. If $f(x) = g(x)$ a.e $x \in E$, we regard $f = g$ as elements of X.

27 **(Definition 6.3)** Let $\{f_k\}_{k \geqq 1} \cup \{f\} \subset L^p(E), E \in \mathscr{M}$. What does it mean if we say that f_k converges to f in L^p? We denote it as $f_k(x) \xrightarrow{L^p} f(x)$.

28 **(Definition 6.4)** Let (X, d) be a metric space where $X \overset{\text{def}}{=} L^p(E), E \in \mathscr{M}$ and $d(f, g) \overset{\text{def}}{=} \|f - g\|_p$. What does it mean if we say that $\{f_k\}_{k \geqq 1} \subset X$ is a Cauchy sequence on (X, d)?

29 **(Theorem 6.5)** Let (X, d) be a metric space where $X \overset{\text{def}}{=} L^p(E)$ and $d(f, g) \overset{\text{def}}{=} \|f - g\|_p$. Show that (X, d) is a complete metric space.

30 **(Exercise 1)** Let $\{f_k(x)\} \subset L^p(E)$ and suppose that $p \geqq 1$. Suppose

$$\|f_{k+1} - f_k\|_p \leqq \frac{1}{2^k}.$$

Show that there exists $f(x) \in L^p(E)$ s.t

$$f_k(x) \xrightarrow{\text{a.e}} f(x) \text{ on } E.$$

31 **(Exercise 2)** Let $\{f_k(x)\}_{k \geqq 1}$ be a sequence of measurable functions and let $F(x) \in L^p(E),\ p \geqq 1$. Suppose that

$$|f_k(x)| \leqq F(x), \ \lim_{k \to \infty} f_k(x) = f(x) \text{ a.e } x \in E.$$

Show that

$$f_k(x) \xrightarrow{L^p} f(x).$$

32 **(Exercise 3)** Let $\{f_n(x)\}_{n\geqq 1} \subset L^2(E), E \in \mathscr{M}$ and suppose that $f_n(x) \xrightarrow{\text{a.e}} f(x)$ on E and $\|f_n\|_2 \leqq M < \infty$. Discuss if $f_n(x)$ converges to $f(x)$ in L^2.

33 **(Exercise 4)** Let us consider equivalent classes in $L^p(E) mE \in \mathscr{M}$. (If $f, g \in L^p(E)$ satisfies $f = g$ a.e $x \in E$, we consider that $f \sim g$.)

(1) Show that each class contains at most one continuous function defined on E.

(2) Show that there exists a class which does not contain any continuous functions.

34 **(Exercise 5)** Let $1 \leqq q \leqq p < \infty$ and let $E \in \mathscr{M}$ with $m(E) < \infty$. Suppose that

$$\lim_{k\to\infty} \int_E |f_k(x) - f(x)|^p dx = 0.$$

Show that

$$\lim_{k\to\infty} \int_E |f_k(x) - f(x)|^q dx = 0.$$

35 **(Exercise 6)** Let $\{f_k(x)\} \cup \{f(x)\} \subset L^p([a,b])$ where $p \geqq 1$. Suppose that $f_k(x) \xrightarrow{L^p} f(x)$. Show that

$$\lim_{k\to\infty} \int_a^x f_k(t)dt = \int_a^x f(t)dt \text{ for all } x \in [a,b].$$

36 **(Exercise 7)** Let $\{f_k(x)\} \cup \{f(x)\} \in L^p(E), E \in \mathscr{M}$ and let $\{g_k(x)\} \cup \{g(x)\} \subset L^q(E)$ where $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$. Suppose that $f_k(x) \xrightarrow{L^p} f(x)$, $g_k(x) \xrightarrow{L^q} g(x)$. Show that

$$\lim_{k\to\infty} \int_E |f_k(x)g_k(x) - f(x)g(x)| dx = 0.$$

(II) $L^p(E)$ as a separable metric space

37 **(Definition 6.5)** Let (X, d) be a metric space where $X = L^p(E), E \in \mathscr{M}$, $d(f,g) \overset{\text{def}}{=} \|f - g\|_p$. Let $\Gamma \subset X$.

(1) What does it mean if we say that Γ is a dense subset of X?

(2) What doest it mean if we say that (X, d) is separable (or a separable metric space)?

38 **(Lemma 6.6)** Let $f(x) \in L^p(E)$ where $E \subset \mathbb{R}^d, E \in \mathscr{M}, 1 \leqq p < \infty$. Let $\epsilon > 0$ be an arbitrary positive number.

(1) Show that there exists a continuous function $g(x)$ defined on $\mathbb{R}^d$ with a compact support s.t

$$\int_E |f(x) - g(x)|^p dx < \epsilon.$$

(2) Show that there exists a step function $\varphi(x) = \sum_{i=1}^{k} c_i \chi_{I_i}(x)$ defined on $\mathbb{R}^d$ with a compact support where $\{I_i\}_{i=1}^{k}$ are rectangles s.t

$$\int_E |f(x) - \varphi(x)|^p dx < \epsilon.$$

39 **(Theorem 6.7)** Show that (X, d) is a seperable metric space where $X = L^p(E), E \in \mathscr{M}, d(f, g) \stackrel{\text{def}}{=} \|f - g\|_p$.

40 **(Corollary 6.8)** Let $1 \leqq p < \infty, 1 \leqq r \leqq \infty$. Show that $L^p(E) \cap L^r(E)$ is dense in $L^p(E)$.

41 **(Theorem 6.9)** Let $f(x) \in L^p(\mathbb{R}^d)$ where $1 \leqq p < \infty$. Show that

$$\lim_{t \to} \int_{\mathbb{R}^d} |f(x+t) - f(x)|^p dx = 0.$$

42 **(Example 1)** Let $f(x) \in L^p(\mathbb{R}^d)$ where $1 \leqq p < \infty$. Show that

$$\lim_{t \to} \int_{\mathbb{R}^d} |f(x) + f(x-t)|^p dx = 2 \int_{\mathbb{R}^d} |f(x)|^p dx.$$

43 **(Example 2)** Let $f(x)$ be Lebesgue measurable on $\mathbb{R}^d$. Show that $f(x)$ is measurable on any $E \subset \mathbb{R}^d, E \in \mathscr{M}$ with $m(E) < \infty$ if and only if there exists $f_1(x) \in L^1(\mathbb{R}^d)$, $f_2(x) \in L^\infty(\mathbb{R}^d)$ s.t $f(x) = f_1(x) + f_2(x)$.

44 **(Exercise 1)** Let $1 < p < \infty$ and let $\{f_n(x)\}_{n \geqq 1} \cup \{f(x)\} \subset L^p(\mathbb{R})$ with $\sup_{n \geqq 1} \|f_n\|_p \leqq M < \infty$. Suppose that

$$\lim_{n \to \infty} \int_0^x f_n(t) dt = \int_0^x f(t) dt, \ x \in \mathbb{R}.$$

Show that for all $g(x) \in L^q(\mathbb{R})$ where $\frac{1}{p} + \frac{1}{q} = 1$,

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n(x) g(x) dx = \int_{\mathbb{R}} f(x) g(x) dx.$$

45 **(Exercise 2)** Show that $L^\infty((0, \infty))$ is not separable. Hint. Consider $f_t(x) \stackrel{\text{def}}{=} \chi_{(0,t)}(x)$ where $0 < t < 1$.

§ 6.3 $L^2(E)$ as an inner product space

(I) inner product and orthogonal system

First, let us recall the definition of a vector space. Let $X \stackrel{\text{def}}{=} L^2(E), E \in \mathscr{M}$. Let $f, g \in X$. We define $(f+g)(t) \stackrel{\text{def}}{=} f(t) + g(t)$, $(\alpha f)(t) = \alpha \cdot f(t)$ $(\alpha \in \mathbb{R})$. Then $f + g \in X, \alpha f \in X$. And if $f(t) = g(t)$ a.e $x \in E$, then we regard f and g are equivalent as elements of X and denote $f \stackrel{X}{=} g$. Then we can regard X as a vector space because

- $f+g \overset{X}{=} g+f$ for all $f,g \in X$.
- $(f+g)+h \overset{X}{=} f+(g+h)$ for all $f,g,h \in X$.
- $\exists 0 \in X$ s.t $f+0 \overset{X}{=} f$ for all $f \in X$.
- $\forall f \in X$, $\exists g \in X$ s.t $f+g \overset{X}{=} 0$.
- $1f \overset{X}{=} f$ for all $f \in X$.
- $\alpha(\beta f) \overset{X}{=} (\alpha\beta)f$ for all $f \in X$ and $\alpha,\beta \in \mathbb{R}$.
- $(\alpha+\beta)f \overset{X}{=} \alpha f+\beta f$ and $\alpha(f+g) \overset{X}{=} \alpha f+\alpha g$ for all $f \in X$ and $\alpha,\beta \in \mathbb{R}$.

Next, we define an inner product on X.

$$< f,g > \overset{\text{def}}{=} \int_E f(x)g(x)dx, \text{ where } f,g \in X.$$

46 **(Basic)** Answer the following questions.

(1) Does $< \cdot,\cdot >: X\times X \mapsto \overline{\mathbb{R}}$ defined above take $\pm\infty$? (or $< \cdot,\cdot >: X\times X \mapsto \mathbb{R}$?)

(2) Verify that $< \cdot,\cdot >$ defined above is an inner product on $X \overset{\text{def}}{=} L^2(E)$.

47 **(Example 1)** Let $f,g \in L^2(E)$. Show that

$$2\|fg\|_1 \leqq t\|f\|_2^2 + \frac{1}{t}\|g\|_2^2 \ \forall\ t>0.$$

48 **(Example 2)** Let $f(x)$ be a non-negative measurable function defined on $[0,\infty)$. Show that

$$\left(\int_0^\infty f(x)dx\right)^4 dx \leqq \pi^2 \int_0^\infty f(x)^2dx \cdot \int_0^\infty x^2f(x)^2dx.$$

49 **(Example 3)** Let $\mathbb{R}_+^2 \overset{\text{def}}{=} (0,\infty)\times(0,\infty)$ and let $f(x,y)$ be a non-negative measurable function defined on $\mathbb{R}_+^2$. Show that

$$\left(\iint_{\mathbb{R}_+^2} f(x,y)dxdy\right)^4 \leqq C\iint_{\mathbb{R}_+^2} f(x,y)^2dxdy \cdot \iint_{\mathbb{R}_+^2} (x^2+y^2)^2f(x,y)^2dxdy,$$

where $C=\frac{\pi^4}{16}$.

50 **(Example 4)** Let $f(x) \in L^2([0,1])$. Suppose that

$$\int_0^1 x^nf(x)dx = \frac{1}{n+2},\ \forall\ n \in \mathbb{N}.$$

Show that $f(x)=x$ a.e $x \in [0,1]$.

51 **(Theorem 6.10 continuity of inner product)** Let $\{f_k\}_{k\geqq 1} \cup \{f\} \subset L^2(E)$. Show that for all $g \in L^2(E)$, we have

$$\lim_{k\to\infty} < f_k, g >=< f, g > .$$

52 **(Definition 6.6)** Answer the following questions.

(1) Let $f, g \in L^2(E)$. What does it mean if we say that f, g are orthogonal?

(2) Let $\{\varphi_\alpha\}_{\alpha\in I} \subset L^2(E)$. What does it mean if we say that $\{\varphi_\alpha\}_{\alpha\in I}$ are orthogonal systems.

(3) Let $\{\varphi_\alpha\}_{\alpha\in I} \subset L^2(E)$. What does it mean if we say that $\{\varphi_\alpha\}_{\alpha\in I}$ are normalized orthogonal systems.

53 **(Example 5)** Verify that

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \cdots \frac{\cos kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}, \cdots\right\}$$

are normalized orthogonal systems on $L^2([-\pi, +\pi])$

54 **(Theorem 6.11)** Show that any standard orthogonal systems on $L^2(E), E \in \mathscr{M}$ is countable.

55 **(Exercise 1)** Let $f, g \in L^2(E), E \in \mathscr{M}$. Show that

$$\|f+g\|_2^2 + \|f-g\|_2^2 = 2(\|f\|_2^2 + \|g\|_2^2).$$

56 **(Exercise 2)** Suppose that $\|f_n - f\|_2 \to 0$ and $\|g_n - g\|_2 \to 0$ as $n \to \infty$ where $\{f_n\}_{n\geqq 1} \cup \{g_n\}_{n\geqq 1} \cup \{f, g\} \subset L^2(E)$. Show that

$$| < f_n, g_n > - < f, g > | \to 0.$$

57 **(Exercise 3)** Suppose that $\|f\|_2 = \|g\|_2$ where $f, g \in L^2(E)$. Show that

$$< f+g, f-g >= 0.$$

58 **(Exercise 4)** Suppose that $\|f_n\|_2 \to \|f\|_2$ and $< f_n, f >\to \|f\|_2^2$ as $n \to \infty$ where $\{f_n\}_{n\geqq 1} \cup \{f\} \subset L^2(E)$. Show that

$$\|f_n - f\|_2 \to 0$$

(II) Generalized Fourier Series

59 **(Definition 6.7)** Let $\{\varphi_n\}_{n\geqq 1} \subset L^2(E), E \in \mathscr{M}$ be normalized orthogonal systems.

(1) What are generalized Fourier coefficienst? Please explain using $\{\varphi_n\}_{n\geqq 1}$.

(2) What are generalized Fourier series? Please explain using $\{\varphi_n\}_{n\geqq 1}$.

60 **(Theorem 6.12)** Let $\{\varphi_n\}_{n\geqq 1} \subset L^2(E), E \in \mathscr{M}$ be normalized orthogonal systems, and let $f \in L^2(E)$. We define

$$f_k(x) \stackrel{\text{def}}{=} \sum_{i=1}^{k} a_i \varphi_i(x),$$

where $a_i \in \mathbb{R}$ for each $i = 1, 2, \cdots, n$. Show that when $a_i = c_i \stackrel{\text{def}}{=} < f, \varphi_i >$, $\|f - f_k\|_2$ attains the minimum value.

61 **(Theorem 6.13 Bessel's Inequality)** Let $\{\varphi_n\}_{n\geqq 1} \subset L^2(E), E \in \mathscr{M}$ be normalized orthogonal systems, and let $f \in L^2(E)$. Show that the generalized Fourier coefficients $\{c_k\}_{k\geqq 1}$ ($c_k \stackrel{\text{def}}{=} < f, \varphi_k >$) satisfy

$$\sum_{k=1}^{\infty} c_k^2 \leqq \|f\|_2^2.$$

62 **(Theorem 6.14 Riesz-Fischer's Theorem)** Let $\{\varphi_n\}_{n\geqq 1} \subset L^2(E), E \in \mathscr{M}$ be normalized orthogonal systems. Suppose that $\{c_k\}_{k\geqq 1} \subset \mathbb{R}$ satisfies

$$\sum_{k=1}^{\infty} c_k^2 < \infty.$$

Show that there exists $g \in L^2(E)$ s.t

$$< g, \varphi_k >= c_k \text{ for each } k \in \mathbb{N}.$$

63 **(Definition 6.8)** Let $\{\varphi_n\}_{n\geqq 1} \subset L^2(E), E \in \mathscr{M}$ be orthogonal systems. What are complete orthogonal systems? Please explain using $\{\varphi_n\}_{n\geqq 1}$.

64 **(Theorem 6.15)** Let $\{\varphi_n\}_{n\geqq 1} \subset L^2(E), E \in \mathscr{M}$ be complete normalized orthogonal systems, let $f \in L^2(E)$, and let $c_k \stackrel{\text{def}}{=} < f, \varphi_k >$ for each $k \in \mathbb{N}$. Show that

$$\lim_{k\to\infty} \left\| \sum_{i=1}^{k} c_i \varphi_i - f \right\|_2 = 0.$$

65 **(Example 6 trigonometric functions as perfect orthogonal systems)** Let $E \stackrel{\text{def}}{=} [-\pi, \pi]$. Show that $\{\phi_k\} = \{1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots\}$ are complete orthogonal systems of $L^2(E)$.

66 **(Definition 6.9)** Let $\psi_1(x), \cdots \psi_k(x)$ be functions defined on $E \in \mathscr{M}$. What does it mean if we say that $\psi_1(x), \cdots \psi_k(x)$ are linearly independent?

67 **(Example 7)** Explain that orthogonal systems $\{\varphi_k\}_{k\geqq 1} \subset L^2(E)$ are linearly independent.

68 **(Theorem 6.16)** Let $\{\varphi_k\}_{k\geqq 1} \subset L^2(E)$ be normalized orthogonal systems. Let $f \in L^2(E)$ and let $\epsilon > 0$. Show that we can always find a linear combination

$$g(x) \stackrel{\text{def}}{=} \sum_{i=1}^{k} a_i \varphi_{k_i}(x),$$

such that

$$\|f - g\|_2 < \epsilon$$

69 **(Exercise 1)** Show that $\{\sin nx\}_{n\geqq 1} \subset L^2([0,\pi])$ are complete orthogonal systems.

70 **(Exercise 2)** Let $f \in L^1([-\pi,\pi])$ and let $\{\varphi_n\}_{n\geqq 1}$ be $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots\}$. Suppose that

$$\int_{[-\pi,\pi]} f(x)\varphi_n(x)dx = 0 \text{ for each } n \in \mathbb{N}.$$

Show that

$$f(x) = 0 \text{ a.e } x \in [-\pi,\pi].$$

71 **(Exercise 3)** Let $\{\varphi_n\}$ be normalized complete orthogonal systems of $L^2(A), A \in \mathscr{M}$, and let $\{\psi_n\}$ be normalized complete orthogonal systems of $L^2(B), B \in \mathscr{M}$. Show that

$$\{f_{i,j}(x,y)\}_{i,j\in\mathbb{N}} \stackrel{\text{def}}{=} \{\varphi_i(x)\cdot\psi_j(x)\}_{i,j\in\mathbb{N}}$$

are complete orthogonal systems on $L^2(A\times B)$.

72 **(Exercise 4)** Let $\{\varphi_k\}$ be normalized orthogonal systems of $L^2(E)$ and let $f \in L^2(E), E \in \mathscr{M}$. Show that

$$\lim_{k\to\infty} \int_E f(x)\varphi_k(x)dx = 0.$$

73 **(Exercise 5)** Let $\{\varphi_k\}_{k\geqq 1} \subset L^2([a,b])$ be normalized complete orthogonal systems and let $f \in L^2([a,b])$. Let us consider the generalized Fourier series of f with respect to $\{\varphi_k\}_{k\geqq 1}$,

$$\sum_{k=1}^{\infty} c_k\varphi_k(x) \text{ where } c_k \stackrel{\text{def}}{=} < f, \varphi_k > .$$

Let $E \subset [a,b]$ be a Lebesgue measurable set. (i.e $E \in \mathscr{M}$.) Show that

$$\int_E f(x)dx = \sum_{k=1}^{\infty} c_k \int_E \varphi_k(x)dx.$$

§ 6.4 Norm of L^p space and Its Formula

74 **(Theorem 6.17)** Let (p,q) be numbers which satisfy $\frac{1}{p}+\frac{1}{q} = 1$ where $1 \leqq p < \infty$. Let $f(x) \in L^p(E)$. Show that there exists $g(x) \in L^q(E)$ with $\|g\|_q = 1$ s.t

$$\|f\|_p = \int_E f(x)g(x)dx.$$

75 **(Theorem 6.18)** Let $f \in L^\infty(E)$. Show that

$$\|f\|_\infty = \sup_{\|g\|_1=1}\left\{\left|\int_E f(x)g(x)dx\right|\right\}$$

76 **(Theorem 6.19)** Let $g(x)$ be a Lebesgue measurable function defined on $E \subset \mathbb{R}^d$. Suppose that there exists $M > 0$ s.t for any integrable simple function $\varphi : E \mapsto \mathbb{R}$,

$$\left|\int_E g(x)\varphi(x)dx\right| \leqq M\|\varphi\|_p,$$

holds.

(1) Show that $g(x) \in L^q(E)$ where $\frac{1}{p} + \frac{1}{q} = 1$.

(2) Show that $\|g\|_q \leqq M$.

77 **(Theorem 6.20 Generalized Minkovski's Inequality)** Let $f(x, y)$ be a Lebesgue measurable function on $\mathbb{R}^d \times \mathbb{R}^d (= \mathbb{R}^{2d})$. Suppose that for all $y \in \mathbb{R}^d$, $x \mapsto f(x, y) \in L^p(\mathbb{R}^d)$. $(1 \leqq p < \infty$ and suppose that

$$\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(x,y)|^p dx\right)^{1/p} dy = M < \infty.$$

Show that

$$\left(\left|\int_{\mathbb{R}^d} f(x,y)dy\right|^p dx\right)^{1/p} \leqq \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(x,y)|^p dx\right)^{1/p} dy$$

78 **(Notice 1)** Consider the following function on $\mathbb{R} \times [0, 2]$ and derive Theorem 6.3 Minkovski's Inequality by applying Thorem 6.20 the generalized Minkovski's Inequality.

$$f(x,y) \overset{\text{def}}{=} \begin{cases} f(x) & 0 \leqq y < 1 \\ g(x) & 1 \leqq y \leqq 2 \end{cases}$$

79 **(Notice 2)** Consider the following function on $(0, \infty) \times [0, 2]$ and derive the discrete version of Minkovski's Inequality by applying Thorem 6.20 the generalized Minkovski's Inequality.

$$f(x,y) \overset{\text{def}}{=} \begin{cases} a_n & n \leqq x < n < 1 \\ b_n & n \leqq x < n + 1 \end{cases}$$

80 **(Example Hardy's Inequality)** Let $1 < p < \infty$ and let $f(x) \in L^p((0, \infty))$. Let us define the function

$$F(x) \overset{\text{def}}{=} \frac{1}{x}\int_0^x f(t)dt,\ x > 0.$$

(1) Show that $F(x) \in L^p(E)$.

(2) Show that

$$\|F\|_p \leqq \frac{p}{p-1}\|f\|_p.$$

§ 6.5 Convolution

81 **(Theorem 6.21 Young's Inequality)** Let $f(x) \in L^1(\mathbb{R}^d)$ and let $g(x) \in L^p(\mathbb{R}^d)$ where $1 < p < \infty$. Show that

$$\|f * g\|_p \leqq \|f\|_1 \cdot \|g\|_p.$$

*

Suppose that $K(x)$ is a function defined on $\mathbb{R}^d$ and let $\epsilon > 0$ be a given positive number. Let us define the function $K_\epsilon(x) : \mathbb{R}^d \mapsto \mathbb{R}(\text{or } \overline{\mathbb{R}})$ based on $K(x)$ as below,

$$K_\epsilon(x) \stackrel{\text{def}}{=} \epsilon^{-d} K\left(\frac{x}{\epsilon}\right) = \epsilon^{-d} K\left(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon} \cdots, \frac{x_d}{\epsilon}\right).$$

82 **(Example 1)** Let $K(x) \stackrel{\text{def}}{=} \chi_{B(0,1)}(x),\ x \in \mathbb{R}^d$. Find $K_\epsilon(x)$.

83 **(Theorem 6.22)** Let $K(x) \in L(\mathbb{R}^d)$ with $\|K\|_1 = 1$ and let $f(x) \in L^p(\mathbb{R}^d)$ where $1 \leqq p < \infty$. Show that

$$\lim_{\epsilon \to 0} \|K_\epsilon * f - f\|_p = 0.$$

84 **(Theorem 6.23)** Let $C^{(\infty)}(\mathbb{R}^d)$ be the family of infinitely differentiable functions defined on $\mathbb{R}^d$. Let us define the family of functions

$$C \stackrel{\text{def}}{=} \left\{ f(x) \in C^{(\infty)}(\mathbb{R}^d) \mid f(x) \text{ has a compact support. } \right\}$$

Show that C is dense in $L^p(\mathbb{R}^d)$.

85 **(Theorem 6.24 Urysohn's Theorem)** Let $F \subset \mathbb{R}^d$ be a compact set and let G be an open set with $F \subset G$. Show that there exists $f(x) \in C^{(\infty)}(\mathbb{R}^d)$ with

- $f(x) = 1,\ x \in F$
- $\text{supp}(f) \subset G$
- $0 \leqq f(x) \leqq 1,\ x \in \mathbb{R}^d$.

86 **(Corollary 6.25)** Let $p > 1, \epsilon > 0, M > 0, k_0 \in \mathbb{N}$. Show that there exists $\varphi(x) \in C^{(\infty)}(\mathbb{R}^d)$ which satisfies $\text{supp}(\varphi) \subset \mathbb{R}^d \setminus B(0, k_0)$ and

$$\int_{\mathbb{R}^d} \varphi(x)dx = 1,\ \|\varphi\|_p < \epsilon,\ 0 \leqq \varphi(x) \leqq M\ (x \in \mathbb{R}^d).$$

87 **(Example 2)** Let $1 < p < \infty$. Let us define a subset of the family of inifinitely differentiable functions,

$$A \stackrel{\text{def}}{=} \{f(x) \in C^{(\infty)}(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} f(x)dx = 0\}.$$

Show that A is dense in $L^p(\mathbb{R}^d)$.

88 **(Example 3)** Let $f(x) \in L^\infty(\mathbb{R})$ and let $f_t(x) \overset{\text{def}}{=} f(x-t)$. Suppose that

$$\lim_{t\to\infty} \|f_t - f\|_\infty = 0.$$

Show that exists a uniformly continuous functon $g(x)$ definied on $\mathbb{R}$ s.t $f(x) = g(x)$ a.e $x \in \mathbb{R}$.

89 **(Example 4)** Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with positive measure. Show that $E - E \overset{\text{def}}{=} \{x_1 - x_2 \mid x_1, x_2 \in E\} \supset (-\delta, \delta)$ for some $\delta > 0$ using convolution.

90 **(Example 5)** Let $\{\varphi_k\} \subset L^2(E)$ be complete orthogonal systems. Show that

$$\sum_{k=1}^{\infty} \|\varphi_k\|_1 = \sum_{k=1}^{\infty} \int_E |\varphi(x)|^1 dx = \infty.$$

§ 6.6 Weak Convergence

Now we introduce another concept of convergence related to $L^p(E)$.

91 **(Definition 6.11)** Let $1 \leqq p, q \leqq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\{f_n(x)\}_{n\geqq 1} \cup \{f(x)\} \subset L^p(E)$, $E \in \mathscr{M}$. What does it mean if we say that $f_n(x)$ converges to $f(x)$ weakly in $L^p(E)$? We denote it as

$$f_n(x) \xrightarrow{w} f(x) \in L^p(E).$$

92 **(Example)** Show that

$$\cos nx \xrightarrow{w} 0 \in L^2([0, 2\pi])$$

93 **(Theorem 6.26)** Let $E \subset \mathbb{R}^d$ and let $E \in \mathscr{M}$ with $m(E) < \infty$. Suppose that $f_n(x) \xrightarrow{w} f(x)$ where $\{f_n(x)\}_{n\geqq 1} \cup \{f(x)\} \subset L^p(E)$. Suppose that $\lim_{n\to\infty} f_n(x) = g(x)$ a.e $x \in E$. Show that $f(x) = g(x)$ a.e $x \in E$.

94 **(Theorem 6.27)** Let $1 \leqq p < \infty$ and let $\{f_n(x)\}_{n\geqq 1} \subset L^p(E)$. Suppose that $f_n(x) \xrightarrow{w} f(x) \in L^part(E)$.

(1) Show that

$$\liminf_{n\to\infty} \|f_n\|_p \geqq \|f\|_p.$$

(2) Let us consider the case of $p = \infty$. Moreover we suppose that $m(E) < \infty$. Can we obtain the same inequality?

95 **(Theorem 6.28)** Let $1 < p \leqq \infty$ and let $\{f_n(x)\}_{n\geqq 1} \subset L^p(E)$. Suppose that there exists $M > 0$ s.t $\|f_n\|_p \leqq M < \infty$ for all $n \in \mathbb{N}$. Show that there exists a subsequence n_k s.t

$$f_{n_k}(x) \xrightarrow{w} f(x) \in L^p(E).$$

96 **(Theorem 6.29 Radon's Theorem)** Let $1 < p < \infty$ and let $\{f_n(x)\}_{n \geqq 1} \subset L^p(E)$. Suppose that $f_n(x) \xrightarrow{w} f(x) \in L^p(E)$ and suppose that $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$. Show that

$$f_n(x) \xrightarrow{L^p} f(x).$$

§ 6.7 Exercises

97 **(Exercise 1)** Let $f(x) \in L^\infty(E)$ and let $w(x) > 0$ and suppose that $\int_E w(x)dx = 1$. Show that

$$\lim_{p\to\infty} \left(\int_E |f(x)|^p w(x) dx \right)^{1/p} = \|f\|_\infty.$$

98 **(Exercise 2)** Let $g(x)$ be a Lebesgue measurable funtion defined on $E \subset \mathbb{R}, E \in \mathscr{M}$. Suppose that $\forall f(x) \in L^2(E)$, we have $\|gf\|_2 \leqq M\|f\|_2$. Show that

$$|g(x)| \leqq M < \infty \text{ a.e } x \in E.$$

99 **(Exercise 3)** Let $f(x) > 0$ for all $x \in (0,\infty)$ and suppose that $f(x)$ is integrable on $(0,\infty)$. Let us pick $r \in (1,\infty)$ and let $E \subset (0,\infty), E \in \mathscr{M}$ with $m(E) > 0$. Show that

$$\left(\frac{1}{m(E)} \int_E f(x) dx \right)^{-1} \leqq \left(\frac{1}{m(E)} \int_E \frac{1}{f(x)^r} dx \right)^{1/r}.$$

100 **(Exercise 4)** Let $f(x) \in L^2([0,1])$ and let $g(x) \stackrel{\text{def}}{=} \int_0^1 \frac{f(t)}{|x-t|^{1/2}} dt$ $x \in (0,1)$. Show that

$$\left(\int_0^1 g(x)^2 dx \right)^{1/2} \leqq 2\sqrt{2} \left(\int_0^1 f(x)^2 dx \right)^{1/2}.$$

101 **(Exercise 5)** Show that the following two equalities cannot hold simultaneously.

$$\int_0^\pi (f(x) - \sin x)^2 dx \leqq \frac{4}{9},$$

and

$$\int_0^\pi (f(x) - \cos x)^2 dx \leqq \frac{1}{9}.$$

102 **(Exercise 6)** Let $f(x) \in L^p(\mathbb{R})$ $(p > 1)$ and suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Let $F(x) \stackrel{\text{def}}{=} \int_0^x f(t)dt$ where $x \in \mathbb{R}$. Show that

$$|F(x+h) - F(x)| \sim o(|h|^{1/q}) \text{ as } h \to 0.$$

103 **(Exercise 7)** Let $m(E_k) > 0$ for all $k \in \mathbb{N}$. Suppose that $m(E_k) \to 0$ as $k \to \infty$. Let

$$g_k(x) \stackrel{\text{def}}{=} \frac{\chi_{E_k}(x)}{m(E_k)^{1/q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1, p, q > 1$. Show that for every $f(x) \in L^p(\mathbb{R}^d)$, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} g_k(x) f(x) dx = 0.$$

104 **(Exercise 8)** Let $f(x), g(x) \in L^3(E)$ and suppose that

$$\|f\|_3 = \|g\|_3 = \int_E f^2(x)g(x)dx = 1.$$

Show that

$$g(x) = |f(x)| \text{ a.e } x \in E.$$

105 **(Exercise 9)** Let $f_1(y,z), f_2(x,z), f_3(x,y)$ be non-negative measurable functions defined on $\mathbb{R}^2$. Let $I_i \stackrel{\text{def}}{=} \|f_i\|_2^2$. Let $F(x,y,z) \stackrel{\text{def}}{=} f_1(y,z)f_2(x,z)f_3(x,y)$. Show that

$$\int_{\mathbb{R}^3} F(x,y,z)dxdydz \leqq (I_1 \cdot I_2 \cdot I_3)^{1/2}.$$

106 **(Exercise 10)** Let $f(x) \in L^p(\mathbb{R})$ where $1 \leqq p < \infty$. Let $r, s > 0$ with $r + s = p$. Let $f_h(x) \stackrel{\text{def}}{=} f(x+h)$. Show that

$$\lim_{|h|\to\infty} \|f_h^r f^s\|_1 = 0.$$

107 **(Exercise 11)** Let $f_n(x)$ be absolutely continuous functions defined on $[0,1]$ with $f_n(0) = 0$. Suppose that $\{f_n'(x)\}_{n\geqq 1}$ be a Cauchy sequence on $L^1([0,1])$. ($\lim_{m,n\to\infty} \|f_m' - f_n'\|_1 = 0$.) Show that there exists an absolutely continuous function $f(x)$ defined on $[0,1]$ with $f_n(x) \xrightarrow{u} f(x)$.

108 **(Exercise 12)** Let $E \subset \mathbb{R}^d, E \in \mathscr{M}$. Suppose that $\|f_k - f\|_1 \to 0$, $\|g_k - g\|_1 \to 0$ as $k \to \infty$ on E.

109 **(Exercise 13)** Let $f_k(x) \in L^p([a,b])$ where $1 \leqq p \leqq \infty$. Suppose that

$$\sum_{k=1}^{\infty} \|f_k\|_p < \infty.$$

Show there exists $f(x) \in L^p([a,b])$ s.t

$$\sum_{k=1}^{\infty} f_k(x) = f(x) \text{ a.e } x \in [a,b],$$

and

$$\sum_{k=1}^{\infty} f_k(x) \xrightarrow{L^p} f(x).$$

110 **(Exercise 14)** Let $\{f_k(x)\}_{k\geqq 1} \cup \{f(x)\} \in L^p(E)$ and suppose that

$$\|f_k - f\|_p < \frac{1}{4^{-k/p}}.$$

Show that for all $\delta > 0$, there exists $E_\delta \subset E, E_\delta \in \mathscr{M}$ with $m(E_\delta) < \delta$ s.t

$$f_k(x) \xrightarrow{u} f(x) \text{ on } E \setminus E_\delta.$$

111 **(Exercise 15)** Let $\{\varphi_k(x)\}_{k \geqq 1} \subset L^2(E)$ be complete normalized orthogonal systems. Show that for all $f, g \in L^2(E)$ we have

$$< f, g >= \sum_{k=1}^{\infty} < f, \varphi_k >< g, \varphi_k >$$

112 **(Exercise 16)** Let $\{\varphi_k\} \subset L^2([a, b])$ be complete normalized orthogonal systems. Let $\{\psi_k\} \subset L^2([a, b])$ be orthogonal systems s.t

$$\sum_{n=1}^{\infty} \int_a^b (\varphi_n(x) - \psi_n(x))^2 dx < 1.$$

Show that $\{\psi_k\}$ are complete orthogonal systems in $L^2([a, b])$.

113 **(Exercise 17)** Let $\{\varphi_k\} \subset L^2(E)$ be normalized orthogonal systems and let $\Phi \in L^2(E)$ with

$$|\varphi_k(x)| \leqq |\Phi(x)| \text{ a.e } x \in E.$$

Show that if $\sum_{k=1}^{\infty} a_k \varphi_k(x)$ converges a.e $x \in E$, then $a_k \to 0$ as $k \to \infty$.

Part II

Solutions

CHAPTER 1

Solutions

§ 1.1

1 (Definition 1.17, 1.18, 1.19, 1.20, 1.21)

(1) $\text{diam}(E) = \sup_{x,y \in E}\{|x-y|\}$

(2) $\text{diam}(E) < \infty$

(3) $B(x_0, \delta) = \{x \in \mathbb{R}^d : |x - x_0| < \delta\}$, $C(x_0, \delta) = \{x \in \mathbb{R}^d : |x - x_0| \leqq \delta\}$, where $|x| \overset{\text{def}}{=} (\sum_{i=1}^n x_i^2)^{1/2}$.

(4) An open rectangle is defined as $\prod_{i=1}^d (a_i, b_i)$. A closed rectangle is defined as $\prod_{i=1}^d [a_i, b_i]$. And a half-open rectangle is defined as $\prod_{i=1}^d (a_i, b_i]$ or $\prod_{i=1}^d [a_i, b_i)$.

(5) $\lim_{k\to\infty} |x_k - x| = 0$.

□

2 (Definition 1.21, 1.22, 1.23, 1.24, 1.25)

(1) Let $\{x_n\} \subset E, x_i \neq x_j (i \neq j)$. Suppose $x_n \to x$ as $n \to \infty$. Then x is an accumulation point of E. Let E' be a set of accumulation points of E. Let $\bar{E} = E \cup E'$ be the closure of E.

(2) Let $x \in E$. Suppose $\exists \delta > 0$ s.t $B(x, \delta) \cap E \setminus \{x\} = \emptyset$. Then x is an isolated point of E. We prove that a set of isolated points is expressed as $E \setminus E'$. Let $S \overset{\text{def}}{=} \{x \in E \mid \exists \delta > 0 \text{ s.t } B(x, \delta) \setminus \{x\} = \emptyset\}$. We show that $S = E \setminus E'$. ($\Leftrightarrow S \subset E \setminus E'$ and $S \supset E \setminus E'$.)

STEP 1. ($S \subset E \setminus E'$) Let $x \in S$. Obviously $x \in E$. By definition of S, there is no sequence $\{x_n\}_{n \geqq 1} \subset E$ s.t $x_n \to x$ ($x_i \neq x_j$ if $i \neq j$) because when n is sufficiently large ($n > N$), $x_n \in B(x, \delta)$, hence $x_n = x$ for all $n > N$. ($\because B(x, \delta) \cap E \setminus \{x\} = \emptyset$) This contradicts to the assumption that $x_i \neq x_j$ if $i \neq j$.

STEP 2. $(S \supset E \setminus E')$ We show that $E \setminus S \subset E'$. Let $x \in E \setminus S$. Since $E \setminus S = \{x \in E \mid \forall\delta > 0, B(x,\delta) \cap E \setminus \{x\} \neq \emptyset\}$, this implies that we can find a sequence $\{x_n\}_{n\geqq 1} \subset E$ s.t $x_n \to x$. ($x_i \neq x_j$ if $i \neq j$.) (We can consider $\delta_n > 0$ s.t $\delta_n \searrow 0$ and $x_n \in B(x,\delta_n) \cap E \setminus \{x\}$. Moreover $\delta_{n+1} < |x_n - x|$. Then we can assure that $\{x_n\}_{n\geqq 1}$ are different points from each other.) So $x \in E'$.

(3) E is a closed set meas that $E' \subset E$. (Different books use different definition. We use this definition.)

(4) E^c is closed. Then E is open.

(5) x is an interior point of E means that $\exists\delta > 0$ s.t $B(x,\delta) \subset E$. Let $\mathring{E}$ be a set of interior points of E.

(6) Let $\partial E = \bar{E} \setminus \mathring{E}$ be a boundary of E. We prove that $\partial E = A$.

STEP 1. $(\partial E \subset A)$ Let $x \in \partial E$. Since $x \notin \mathring{E}$, $\forall\delta > 0$, $B(x,\delta) \not\subset E$. This implies that $B(x,\delta) \cap E^c \neq \emptyset$ for all $\delta > 0$. Furthermore, $x \in \overline{E} = E \cup E'$. We consider the cases $x \in E$ and $x \in E'$.

case 1. $(x \in E)$ Obviously $\forall\delta > 0$, $B(x,\delta) \cap E \neq \emptyset$ because $B(x,\delta) \cap E$ contains x.

case 2. $(x \in E')$ There exists $\{x_n\}_{n\geqq 1} \subset E$ s.t $x_n \to x$ ($x_i \neq x_j$ if $i \neq j$.) From this fact, we find out that $\forall\delta > 0$, $B(x,\delta) \cap E \neq \emptyset$ because for sufficiently large n, $|x_n - x| < \delta$, so $B(x,\delta) \cap E$ contains $\{x_n\}_{n\geqq N_\delta}$ where N_δ is a sufficiently large natural number.

STEP 2. $(\partial E \supset A)$ Let $x \in A$. Since $\forall\delta > 0$, $B(x,\delta) \cap E^c \neq \emptyset$, (so $B(x,\delta) \not\subset E$), so x is not an interior point of E. So $x \notin \mathring{E}$. Since $\forall\delta > 0$, $B(x,\delta) \cap E \neq \emptyset$, we have $x \in \overline{E}$. The reason is as below. If $x \in E$, the statement holds obviously. So we suppose $x \notin E$. Then we can pick $x_n \in B(x,\delta_n) \cap E$ with $\delta_n \to 0$. So $x_n \to x$. We may suppose that $x_i \neq x_j$ if $i \neq j$ because we can take a subsequence $n_{(k)}$ so that $0 < |x_{n_{(k+1)}} - x| < |x_{n_{(k)}} - x|$. So $x \in E'$. From this argument, we conslude that $x \in E \cup E'$.

□

3 (Theorem 1.13)

STEP 1. $(\Rightarrow)$ Since $x \in E'$, we have $\{x_n\}_{n\geqq 1}(i \neq j \Rightarrow x_i \neq x_j) \subset E, x_n \to x$. For any $\delta > 0$, since $x_n \to x$, we can find N s.t $|x - x_n| < \delta(n > N)$. Therefore $B(x.\delta) \cap E \setminus \{x\} \supset \{x_n\}_{n>N}$.

STEP 2. $(\Leftarrow)$ We consider $\{\delta_n\}_{n\geq 1}$ s.t $\delta_n \searrow 0$. We pick $x_1 \in B(x,\delta_1) \cap E \setminus \{x\}$. Next we pick $x_2 \in B(x,\delta_2) \cap E \setminus \{x\}$. (But we assume that $|x - x_2| < |x - x_1|$ to assure that $\{x_n\}$ are different from each other.) In this way we obtain $\{x_n\}_{n\geqq 1} \subset E(i \neq j \Rightarrow x_i \neq x_j)$ s.t $x_n \to x$. So $x \in E'$.

□

4 (Theorem 1.14)

STEP 1. ($(E_1 \cup E_2)' \supset E_1' \cup E_2'$) Since $E_1 \subset E_1 \cup E_2$, we have $E_1' \subset (E_1 \cup E_2)'$. (Because if $\exists \{x_n\}_{n \geqq 1} \subset E_1$ s.t $x_n \to x$, we can say that $\exists \{x_n\}_{n \geqq 1} \subset E_1 \cup E_2$ s.t $x_n \to x$.) Similarly we have $E_2' \subset (E_1 \cup E_2)'$. So we have the desired result.

STEP 2. ($(E_1 \cup E_2)' \subset E_1' \cup E_2'$) Let $x \in (E_1 \cup E_2)'$. Let $\{x_n\}_{n \geqq 1} \subset E_1 \cup E_2$ where $x_i \neq x_j (i \neq j)$ and $x_n \to x$. Since $\{x_n\}_{n \geqq 1} \subset E_1 \cup E_2$, E_1 (or E_2) contain infinitely many $\{x_n\}$. We can choose infinitely many $\{x_n\} \subset E_1$. Hence we have a subsequence n_k s.t $\{x_{n_k}\}_{k \geqq 1} \subset E_1$. Of course, $x_{n_k} \to x$. Hence $x \in E_1'$. From this discussion, x is always containd in either E_1' or E_2'. So $x \in E_1' \cup E_2'$. Now we have the desired result.

□

5 (Theorem 1.15 Bolzano-Weierstrass Theorem on $\mathbb{R}^d$) Let $\{x_{k,1}\}_{k \geqq 1} \subset E$ where $x_k = (x_{k,1}, x_{n,2}, \cdots, x_{k,d})^T$. Since E is bounded, $|x_{k,1}| \leqq M_1 < \infty, |x_{k,2}| \leqq M_2 \cdots |x_{k,d}| \leqq M_d$ for some $M_1, \cdots M_d < \infty$. By Bolzano Weierstrass's theorem for $\mathbb{R}^1$, we can find a subsequence $\{k_1(\ell)\}_{\ell \geqq 1}$ s.t $x_{k_1(\ell),1}$ converges to some $x_1^* \in \mathbb{R}$. Next, $\{x_{k_1(\ell),2}\}$ is bounded, similarly we can find a subsubsequence $\{k_2(\ell)\} \subset \{k_1(\ell)\}$ s.t $x_{k_2(\ell),2}$ converges to some $x_2^* \in \mathbb{R}$. Of course, $x_{k_2(\ell),1}$ also converges x_1^*. By repeating this process, we will finally obtain $\{x_{k_d(\ell)}\}_{\ell \geqq 1}$ s.t $x_{k_d(\ell)} \to (x_1^*, x_2^* \cdots, x_d^*)^T$. □

6 (Theorem 1.15 Supplement) □

7 (Exercise 1.4.1) $E = \bigcup_{n \in \mathbb{Z}} [n, n+1) \cap E$. Since E is uncountable, there exists $n_0 \in \mathbb{Z}$ s.t $[n_0, n_0 + 1) \cap E$ is an infinite set. (Otherwise E is counable.) Since $[n_0, n_0 + 1) \cap E \subset [n_0, n_0 + 1]$ is bouned, so it has at least one limit point by Bolzano-Weierstrass theorem. So $E' \neq \emptyset$. Now the proof is complete.

We present an alternative solution. We prove the contraposition, that is if $E' = \emptyset$ then E is not uncountable. (At most countable) Note that $E = (E \setminus E') \cup (E \cap E') = E \setminus E'$ and $E \setminus E'$ is a set of isolated points. A set of isolated points is countable. (See Exercise 1.42) Now the proof is complete. □

8 (Exercise 1.4.2)

STEP 1. $E = E \setminus E' \cup E \cap E'$. Since $E \cap E' \subset E'$ is countable, it is enough for us to prove that $E \setminus E'$ is countable. $S \overset{\text{def}}{=} E \setminus E'$. Every point in S is an isolated point. We show that if S is a set of isolated points then S is countable.

STEP 2. Let $S_n \overset{\text{def}}{=} \left\{x \in [-n, n]^d \mid B\left(x, \frac{1}{n}\right) \cap S = \{x\}\right\}$. We claim that $S = \bigcup_{n=1}^{\infty} S_n$.

First, we prove $S \subset \bigcup_{n=1}^{\infty} S_n$. Let $x \in S$. Then there exists sufficiently large $n_1 \in \mathbb{N}$ s.t $x \in [-n_1, n_1]^d$. There also exists sufficiently large $n_2 \in \mathbb{N}$ s.t $B(x, \frac{1}{n_2}) \cap S = \{x\}$. Let $n_0 \overset{\text{def}}{=} \max\{n_1, n_2\}$. Then $x \in S_{n_0}$.

Next, we prove $S \supset \bigcup_{n=1}^{\infty} S_n$. However $S_n \subset S$ holds obviously for all $n \in \mathbb{N}$.

STEP 3. We claim that S_n is a finite set for every $n \in \mathbb{N}$. $\forall x_1, x_2 \in S_n, (x_1 \neq x_2)$, $B(x_1, \frac{1}{n}) \cap B(x_2, \frac{1}{n}) = \emptyset$. Suppose that S_n is infinite, there exists inifinitely many disjoint open balls $\{B(x_k, \frac{1}{n})\}_{k \geqq 1}$ s.t $B(x_k, \frac{1}{n}) \subset [-n - \frac{1}{n}, n + \frac{1}{n}]^d$. However this can not happen because $[-n-\frac{1}{n}, n+\frac{1}{n}]^d$ is bounded and its volume is finite (so it can not contain infinitely many disjoint open balls whose radius is $\frac{1}{n}$). So we conclude that S_n is finite hence S is countable.

□

9 (Exercise 1.4.5) Every point in E is an isolate point. We have alreday proven that a set of isolated points is a countable set in the previous question. □

§ 1.2

10 (Example 2 and 6)

STEP 1. ($\Rightarrow$) Suppose $f(x) \in C(\mathbb{R}^n)$. It is enough for us to show that E_1 is closed for all $t \in \mathbb{R}$. (Here we may fix $t \in \mathbb{R}$.) When $E_1' = \emptyset$, E_1 is closed. So we suppose that $E_1' \neq \emptyset$. Let us pick $x_0 \in E_1'$ and $\{x_n\}_{n \geqq 1} \subset E_1 (i \neq j \Rightarrow x_i \neq x_j)$ s.t $x_n \to x_0$. Then $f(x_n) \geqq t$. By taking $n \to \infty$, $\lim_{n\to\infty} f(x_n) \geqq t$. The left hand side will be $f(x_0)$ because $f(x) \in C(\mathbb{R}^n)$. So $x_0 \in E_1$. This implies $E_1' \subset E_1$ for all $t \in \mathbb{R}$. Therefore E_1 is closed. Similarly E_2 is closed for all $t \in \mathbb{R}$.

STEP 2. ($\Leftarrow$) We prove contraposition of the statement. We show $f(x) \notin C(\mathbb{R}^n) \Rightarrow \exists t \in \mathbb{R}$ s.t E_1 or E_2 is not closed. Now $f(x)$ is not continuous, so $\exists x_0 \in \mathbb{R}^n$ and $\exists \epsilon > 0$ s.t $\forall \delta > 0$ $\exists y \in B(x_0, \delta)$ s.t $|f(y) - f(x_0)| \geqq \epsilon$. This implies we can pick $\{y_n\}_{n\geqq 1} : y_n \to x_0$ s.t $|f(y_n) - f(x_0)| \geqq \epsilon$. (You may consider a decreasing sequence of $\{\delta_n\} : \delta_n \searrow 0$) So $f(y_n) \geqq f(x_0) + \epsilon$ or $f(y_n) \leqq f(x_0) - \epsilon$. At least one of the conditions ($f(y_n) \geqq f(x_0) + \epsilon$ or $f(y_n) \leqq f(x_0) - \epsilon$) holds for infinitely many n. So we can find a subsequence n_k s.t $f(y_{n_k}) \geqq f(x_0) + \epsilon$. Now let $t = f(x_0) + \epsilon$. Then E_1 is not closed because $y_{n_k} \in E_1$ and $y_{n_k} \to x_0$ but $f(x_0) \geqq t$ $(= f(x_0) + \epsilon)$ does not hold. So $x_0 \notin E_1$.

□

11 (Example 3) We show that $\overline{B(x_0, \delta)} = C(x_0, \delta)$. For simplicity, let $B = B(x_0, \delta)$, $C = C(x_0, \delta)$.

STEP 1. From $B \subset C$, we have $\bar{B} = \bar{C}$. Since a closed ball is a closed set, $\bar{C} = C$. So we have $\bar{B} \subset C$.

STEP 2. Next we show $\bar{B} \supset C$. Now let $x \in C$. Let $x_k = (1 - \frac{1}{k})x + \frac{1}{k}x_0$. $|x_k - x| = \frac{1}{k}|x_0 - x| \leqq \frac{\delta}{k} < \delta$, hence $\{x_k\} \subset B$ and $x_k \to x$. Therefore $x \in B' \subset B' \cup B = \bar{B}$.

□

12 (Example 4) Let $\delta > 0$ be an arbitrary small number. Let $m \in \mathbb{N}$ s.t $10^{-m} < \delta$. Let us define $c_1, c_2, \cdots$ and $d_1, d_2, \cdots$ for given natural numbers n_1, n_2 as

$$\begin{aligned} n_1 a - [n_1 a] &= 0.c_1 c_2 c_3 \cdots c_m c_{m+1} \cdots \\ n_2 a - [n_2 a] &= 0.d_1 d_2 d_3 \cdots d_m d_{m+1} \cdots , \end{aligned}$$

where $[x] \stackrel{\text{def}}{=} \max\{k \in \mathbb{Z} | k \leqq x\}$. We can find $n_1, n_2 \in \mathbb{N}(n_1 \neq n_2)$ such that

$$c_1 = d_1, c_2 = d_2, \cdots, c_m = d_m,$$

because the combinations of $\{c_1, c_2 \cdots c_n\}$ have only 10^m but there exists infinitely many natural numbers $(n_1, n_2) \in \mathbb{N}^2$. Moreover since $a \notin \mathbb{Q}$ (an irrational number),

$$|(n_1 a - [n_1 a]) - (n_2 a - [n_2 a])| > 0, \ (n_1 \neq n_2).$$

(if $= 0$, a will be a rational number.) From this fact, we find that we can pick $n_1, n_2 \in \mathbb{N}, k \in \mathbb{Z}$ s.t

$$0 < |n_1 a - n_2 a - k| < 10^{-m} < \delta.$$

Hence we can find $p + aq \in E_a$ s.t $p + aq \in (0, \delta)$. Now for any $x \in \mathbb{R}$, we can find $z \in \mathbb{Z}$ s.t $x - \delta < z(p + aq) < x + \delta$ (because $p + aq$ is very small). This implies that we can find a sequence $\{x_n\} \subset E_a$ s.t $x_n \to x$. □

13 (Example 5)

STEP 1. Since $\cos(x)$ is a continuous function, $\forall x \in \mathbb{R}$, $\exists \delta > 0$ such that $\forall y \in (x - \delta, x + \delta), |\cos(x) - \cos(y)| < \epsilon$. Now choose an arbitrary number $c \in [-1, 1]$. We can find $x \in \mathbb{R}$ s.t $c = \cos(x)$. Now fix $x \in \mathbb{R}$ and $\epsilon > 0$.

STEP 2. Let us be careful of the fact that $E = \{\cos(n) | n \in \mathbb{N}\} = \{\cos(n + 2m\pi) | m, n \in \mathbb{N}\}$. Let $Y = \{m + (2\pi)n | m, n \in \mathbb{N}\}$ (Let us recall the previous exercise). From the previous exercise, we can find $y = m + 2n\pi \in Y$ s.t $|x - y| < \delta$. ($\because 2\pi \notin \mathbb{Q}$). Hence $|\cos(x) - \cos(y)| < \epsilon$. Now $\cos(y) = \cos(m + 2n\pi) = \cos(n)$, therefore we may conclude that $\forall c \in [-1, 1], \exists n$ s.t $|\cos(n) - c| < \epsilon$. This implies that for any $c \in [-1, 1]$, we can find a sequence of natural numbers $\{n_k\}_{k \geqq 1}$ s.t $\lim_{k \to \infty} |\cos(n_k) - c| = 0$. So $c \in \bar{E}$.

□

14 (Theorem 1.16)

(1) We have already shown that $(F_1 \cup F_2)' = F_1' \cup F_2'$. Since F_1, F_2 are closed, $F_1' \cup F_2' \subset F_1 \cup F_2$. Therefore $(F_1 \cup F_2)' \subset F_1 \cup F_2$.

(2) $F \subset F_\alpha (\forall \alpha \in I)$. Hence $F' \subset F_\alpha' = F_\alpha (\forall \alpha \in I)$. Therefore we have $F' \subset \cap_{\alpha \in I} F_\alpha = F$.

□

15 (Theorem 1.17) We consider the following two cases.

case 1. ($F_k \setminus F_{k+1} \neq \emptyset$ for only finite number of k) $\exists k_0 \geqq 1$ such that $F_{k_0+1} = F_{k_0+2} = F_{k_0+3} = \cdots$. Then $\bigcap_{k=1}^{\infty} F_k = F_{k_0} \neq \emptyset$ ($\because$ assumption). So the statement is true.

case 2. ($F_k \setminus F_{k+1} \neq \emptyset$ occurs for inifinitely many k.) We can find a subsequence $F_{k_\ell} \setminus F_{k_\ell + 1} \neq \emptyset$ for all $\ell \in \mathbb{N}$. Let us pick $x_\ell \in F_{k_\ell} \setminus F_{k_\ell + 1}$. Since $\{x_\ell\}_{\ell \geqq 1} \subset F_{k_1} \subset F_1$ and F_1 is bounded and closed, we can find a subsequence $x_{\ell(m)}$ s.t $x_{\ell(m)} \to x^* \in F_1$ by Bolzano-Weierstrass Theorem. And $\{x_{\ell(m)}\}_{m \geqq 2} \subset F_{k_{\ell(2)}} \subset F_2$ and F_2 is closed, so $x^* \in F_2$. By similar argument, we have $x^* \in F_k$ for all $k \in \mathbb{N}$. So $x^* \in \bigcap_{k=1}^{\infty} F_k$.

□

16 (Exercise 1.5.1.4)

STEP 1. ($\supset$) Since $\overline{E} \in \{F\}_{F\supset E;F:\text{closed}}$, $\overline{E} \supset \bigcap_{F\supset E;F:\text{closed}} F$

STEP 2. ($\subset$) Let F be a closed set with $F \supset E$. Then $E' \subset F' \subset F$ so $\overline{E} = E \cap E' \subset F$. Therefore $\overline{E} \subset \bigcap_{F\supset E,F:\text{closed}}$.

□

17 (Exercise 1.5.1.5) Since $f(x)$ is real-valued so $F = \{x \in F \mid f(x) < \infty\} = \bigcup_{n=1}^{\infty}\{x \in F \mid f(x) \leqq n\}$. We prove that for each $n \in \mathbb{N}$, $\{x \in F \mid f(x) \leqq n\}$ is a finite set. Suppose that $F_n \overset{\text{def}}{=} \{x \in F \mid f(x) \leqq n\}$ is not a finite set then we can pick infinitely many points $\{x_k^{(n)}\} \subset F_n$ (if $k \neq \ell$, $x_k^{(n)} \neq x_\ell^{(n)}$). Since $F_n \subset F$ is bounded, we can find a subsequence $\{x_{k(m)}^{(n)}\}_{m\geqq 1}$ s.t $x_{k(m)}^{(n)} \to x_0$ by Bolzano Weierstrass theorem. By assumption $\lim_{m\to\infty} f(x_{k(m)}^{(n)}) = \infty$. This means that we can find m_0 s.t $f(x_{k(m_0)}^{(n)}) > n$. So this contradicts to the assumption. □

18 (Exercise 1.5.1.6) We show that $F' \subset F$. Suppose that $F' \neq \emptyset$. Let $(x_0, y_0) \in F'$. Then $\{(x_n, y_n)\}_{n\geqq 1} \subset F$ s.t $(x_n, y_n) \to (x_0, y_0)$ ($(x_i, y_i) \neq (x_j, y_j)$ if $i \neq j$). For each n, $f(x_n) \geqq y_n$. So $\lim_{n\to\infty} f(x_n) \geqq \lim_{n\to\infty} y_n$. Since $f(x)$ is continuous, $f(x_0) \geqq y_0$. This means that $(x_0, y_0) \in F$. So $F' \subset F$. □

19 (Theorem 1.18)

(1) $G^c = \bigcap_{\alpha\in I} G_\alpha^c$. Since G_α^c are closed sets, G^c is also a closed set. (See Theorem 1.16) So G is an open set.

(2) $(\bigcap_{k\geqq 1}^m G_k)^c = \bigcup_{k\geqq 1}^m G_k^c$ is closed. ($\because$ G_k^c are closed sets. See Theorem 1.16)

(3) Let $F = G^c$.

STEP 1. ($\Rightarrow$) We consider its contraposition. We show that $\exists x \in G, \forall\delta > 0, B(x,\delta)\setminus G \neq \emptyset \Rightarrow G$ is not open (F is not closed). By assumption, by taking a sequence of $\{\delta_n\} : \delta_n \searrow 0$, we may obtain a sequece of point $\{x_n\} \subset B(x, \delta_n) \setminus G = B(x, \delta_n) \cap F$. (Moreover we may assume that $|x - x_{k+1}| < |x - x_k|$. So $x_i \neq x_j$ if $i \neq j$) Since $x_n \to x$, $x \in F'$ but $x \in G$. This implies $F \setminus F' \neq \emptyset$. So F is not closed.

STEP 2. ($\Leftarrow$) We consider its contraposition. We show that G is not open $\Rightarrow$ $\exists x \in G$ s.t $\forall\delta > 0, B(x,\delta) \setminus G \neq \emptyset$. By assumpotion, F is not closed, so there exists $x \in F' \setminus F$. We may take $\{x_n\}_{n\geqq 1} \subset F : x_n \to x \in G(\notin F)$. Then $\forall\delta > 0$, there exists N s.t $\{x_n\}_{n>N} \subset B(x, \delta)$. This implies that $B(x,\delta) \setminus G = B(x,\delta) \cap F \supset \{x_n\}_{n>N} \neq \emptyset$.

□

20 (Example 7) We use the result of the previous problem. We pick $x_0 \in H$. We show that $\exists B(x_0, \delta) \subset H$. By definition,

$$\omega_f(x_0) = \lim_{\delta \searrow 0} \sup_{x_1, x_2 \in B(x_0, \delta)} \{|f(x_1) - f(x_2)|\} < t.$$

Since $\cdots < t$, there exists sufficiently small $\delta_0 > 0$ such that

$$\sup_{x_1, x_2 \in B(x_0, \delta_0)} \{|f(x_1) - f(x_2)|\} < t.$$

We pick an arbitrary point $x^* \in B(x_0, \delta_0)$. Since $B(x_0, \delta_0)$ is an open ball, we may pick $\delta^* > 0$ such that $B(x^*, \delta^*) \subset B(x_0, \delta_0)$. Hence $\sup_{x_1, x_2 \in B(x^* \delta^*)}\{|f(x_1) - f(x_2)|\} \leqq \sup_{x_1, x_2 \in B(x_0, \delta_0)}\{|f(x_1) - f(x_2)|\} < t$. So we have $\lim_{\delta \searrow 0} \omega_f(x^*) < t$ for all $x^* \in B(x_0, \delta_0)$. This implies that $B(x_0, \delta_0) \subset H$. Therefore H is an open set. □

21 (Theorem 1.19)

(1) For each $x \in G$, let $I_x \overset{\text{def}}{=} (a_x, b_x)$ where $a_x \overset{\text{def}}{=} \inf\{a \mid a < x, (a, x) \subset G\}$ and $b_x \overset{\text{def}}{=} \sup\{b \mid b > x, (x, b) \subset G\}$. Since G is an open set, so $I_x \neq \emptyset$.

STEP 1. We prove that $G = \bigcup_{x \in G} I_x$. First, let $x_0 \in G$ be an arbitrary point in G. Then $x_0 \in I_{x_0}$ and $I_{x_0} \subset \bigcup_{x \in G} I_x$. So $G \subset \bigcup_{x \in G} I_x$.

Next, we prove that $I_x \subset G$ for all $x \in G$. Let $x \in G$. We can pick $\{a_n\}$ s.t $a_n \searrow a_x$. Since $(a_n, x) \subset G$ for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty}(a_n, x) \subset G$. The left hand side is $\bigcup_{n=1}^{\infty}(a_n, x) = (a_x x)$. So $(a_x, x) \subset G$. Similarly, $(x, b_x) = \bigcup_{n=1}^{\infty}(x, b_n) \subset G$ where $b_n \nearrow b_x$. So $I_x = (a_x, b_x) \subset G$ for all $x \in G$. Therefore $\bigcup_{x \in G} I_x \subset G$.

STEP 2. We prove that if $x \neq y$ $(x, y \in G)$ then $I_x = I_y$ or $I_x \cap I_y = \emptyset$. Suppose that $I_x \cap I_y \neq \emptyset, x < y$. Since $[x, y] \subset G$, we find out that $a_x = a_y$ and $b_x = b_y$ by their definitions. So G is a union of disjoint open intervals.

STEP 3. Finally, we explain that G is a countable union of disjoint open intervals. Since each disjoint interval contains rational numbers, and the number of rational numbers is countably many, G is a countable union of disjoint open intervals.

(2)

STEP 1. First we prove that G is a countable union of open rectangles. (not disjoint) Let

$$I_{n,k} \overset{\text{def}}{=} \prod_{i=1}^{d} \left(\frac{k_i}{2^n}, \frac{k_i + 1}{2^n} \right], \; n \in \mathbb{N}, k \in \mathbb{Z}^d$$

We claim that

$$G = \bigcup_{n=1}^{\infty} \bigcup_{k \in \mathbb{Z}^d; I_{n,k} \subset G} I_{n,k}.$$

First $\supset$ is obvious because for each $(n, k) \in \mathbb{N} \times \mathbb{Z}^d$, we pick $I_{n,k} \subset G$. Next we prove $\subset$. Let us pick $x \in G$. Since G is an open set, there exists $\delta > 0$ s.t $B(x, \delta) \subset G$. For each $n \in \mathbb{N}$, there always exists $k \in \mathbb{Z}^d$ s.t $x \in I_{n,k}$. By choosing sufficiently large $n \in \mathbb{N}$, we can let $\operatorname{diam}(I_{n,k}) = \frac{\sqrt{\delta}}{2^n} < \delta$. So we have $x \in I_{n,k} \subset B(x, \delta) \subset G$. Such $I_{n,k}$ is contained in the union of the right hand side. So $x \in \bigcup_{n, \in \mathbb{N}} \bigcup_{k \in \mathbb{Z}^d; I_{n,k} \subset G} I_{n,k}$. Now the proof is complete.

STEP 2. Let

$$G_n = \bigcup_{m=1}^{n} \bigcup_{k\in\mathbb{Z}^d; I_{m,k}\subset G} I_{m,k}.$$

Since $H_n \overset{\text{def}}{=} G_n \setminus G_{n-1}$ can be expressed as disjoint union of open rectangles $\{I_{n,k}\}_{k\in\mathbb{Z}^d}$, and $G = \bigcup_{n=1}^{\infty} H_n$, we have the desired conclusion.

□

22 (Exercise 1.5.2.1) Show that $\mathring{E} = \left(\overline{E^c}\right)^c$.

STEP 1. ($\mathring{E} \subset \left(\overline{E^c}\right)^c$) We show that $(\mathring{E})^c \supset \overline{E^c} = (E^c) \cup (E^c)'$. Since $(\mathring{E})^c$ is closed, it is enough for us to show that $(\mathring{E})^c \supset E^c$. However this is obvious because $\mathring{E} \subset E$.

STEP 2. ($\mathring{E} \supset \left(\overline{E^c}\right)^c$) We show that $(\mathring{E})^c \subset \overline{E^c}$. Let $x \in (\mathring{E})^c$. We show that $x \in \overline{E^c}$.

case 1. ($x \in E^c$) $x \in E^c \subset E^c \cup (E^c)' = \overline{E^c}$.

case 2. ($x \notin E^c$) Since $x \notin \mathring{E}$, $\forall\delta > 0$, $B(x, \delta) \not\subset E$. Therefore $\forall\delta > 0$, $B(x, \delta) \cap E^c \neq \emptyset$. Moreover $x \notin E^c$, implies that $\forall\delta > 0$, $B(x, \delta) \setminus \{x\} \cap E^c \neq \emptyset$. So $x \in (E^c)'$. Therefore $x \in (E^c) \cup (E^c)' = \overline{E^c}$.

□

23 (Exercise 1.5.2.3)

(1) Let us recall that $\partial G = \{x \in \mathbb{R}^d \mid \forall\delta > 0, B(x, \delta) \cap G \neq \emptyset, B(x, \delta) \cap G^c \neq \emptyset\}$ from the previous question. From this, it is easy to find out that $\partial G = \partial(G^c)$.

STEP 1. (G is open $\Rightarrow G \cap \partial G = \emptyset$) Let $x \in G$. Then $\exists\delta > 0$ s.t $B(x, \delta) \subset G$. So $B(x, \delta) \cap G^c = \emptyset$. Therefore $x \notin \partial G$. This implies that $G \cap \partial G = \emptyset$.

STEP 2. (G is open $\Leftarrow G \cap \partial G = \emptyset$) Let us pick $x \in G$. Since $x \notin \partial G$, $\exists\delta > 0$ s.t $B(x, \delta) \cap G = \emptyset$ or $B(x, \delta) \cap G^c = \emptyset$ holds. $x \in G$, $\{x\} \in B(x, \delta) \cap G \neq \emptyset$, so $B(x, \delta) \cap G^c = \emptyset$ holds. This implies that $B(x, \delta) \subset G$. So G is an open set.

(2) Let $G \overset{\text{def}}{=} F^c$. Then $\partial G = \partial F$. G is open if and only if F is closed. $\partial F \subset F \Leftrightarrow \partial F \cap F^c = \emptyset \Leftrightarrow \partial G \cap G = \emptyset \Leftrightarrow G$ is open. ($\because$ the previous question.)

□

24 (Exercise 1.5.2.4) Let $a \in A$. There exists $x \in G$ s.t $a \in B(x, r_0)$. Since G is an open set, there exists $\delta > 0$ s.t $B(x, \delta) \subset G$. We may suppose $0 < \delta < r_0 - |x - a|$. ($|x-a| < r_0$) We pick $x^* \in B(x, \delta) \subset G$. Then $|x^*-a| \leqq |x-a|+|x-x^*| < |x-a|+\delta < r_0$. So $a \in B(x^*, r_0) \subset \bigcup_{x\in G} \overline{B}(x, r_0) = A$. This implie that a is an interior point of A. So A is an open set. □

25 (Exercise 1.5.2.5) □

26 (Definition 1.26, Lemma 1.20, Lindelof's Covering Lemma)

(1) Let $E \subset \mathbb{R}^d$. Let $\Gamma = \{G_\alpha\}_{\alpha \in I}$ be a family of open sets on $\mathbb{R}^d$. If $E \subset \bigcup_{\alpha \in I} G_\alpha$, we say that Γ is an open cover of E. If $\Gamma' \subset \Gamma$ is also open cover of E, then Γ' is called a sub cover of Γ.

(2) We fix $x \in E$. We can find $r > 0$ such that $B(x, r) \subset E$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find $y = (y_1, y_2, \cdots, y_d) \in \mathbb{Q}^d$ s.t $|x_i - y_i| \leqq \frac{r}{4\sqrt{d}}$. Then $|x - y| \leqq \frac{r}{4}$. Now we choose $q \in \mathbb{Q} \cap (r/4, r/2)$. Then $x \in B(y, q) \subset B(x, r)$. Such $B(y, q) \in \mathscr{A}$.

(3) For each $x \in E$, we can find at least one $\alpha(x)$ s.t $x \in G_{\alpha(x)}$. We apply the previous lemma to each $G_{\alpha(x)}$. Then we may find $B(y_{(x)}, q_{(x)}) \in \mathscr{A}$ such that $x \in B(y_{(x)}, q_{(x)}) \subset G_{\alpha(x)}$. $E = \bigcup_{x \in E} \{x\} \subset \bigcup_{x \in E} B(y_{(x)}, q_{(x)})$. Since $\{B(y_{(x)}, q_{(x)})\}_{x \in E} \subset \mathscr{A}$ is countable, we may rewrite it as $E \subset \bigcup_{k=1}^{\infty} B(x_k, q_k)$. For each k, we may find $\alpha_k \in I$ s.t $B(x_k, q_k) \subset G_{\alpha_k}$. Therefore $E \subset \bigcup_{k \geqq 1} G_{\alpha_k}$.

□

27 (Theorem 1.21 Heine-Borel's Finite Covering Lemma) Let $F \subset \mathbb{R}^d$ be a closed and bounded set. Suppose that there exists an open cover $\{G_\alpha\}_{\alpha \in I}$ (I is an index set. I can be countable or uncountable.) Then we can find a finite cover $\{\alpha_1, \alpha_2, \cdots, \alpha_m\} \subset I$ s.t

$$E \subset \bigcup_{k=1}^{n} G_{\alpha_k}.$$

This is called Heine-Borel's Finite Covering Lemma.

STEP 1. By Lemma 1.20 Lindelof's Covering Lemma, we may suppose that

$$F \subset \bigcup_{n=1}^{\infty} G_n$$

without loss of generality.

STEP 2. Let

$$H_n \stackrel{\text{def}}{=} \bigcup_{k=1}^{n} G_k, \ L_n \stackrel{\text{def}}{=} F \setminus \bigcup_{k=1}^{n} G_k$$

We consider the following two cases.

case 1. ($L_n = \emptyset$ for some $n \in \mathbb{N}$) This implies that $F \subset \bigcup_{k=1}^{n} G_n$ for some $n \in \mathbb{N}$. So the theorem is true for this case.

case 2. ($L_n \neq \emptyset$ for all $n \in \mathbb{N}$) Note that L_n is a bounded closed set for each $n \in \mathbb{N}$, and $L_n \supset L_{n+1}$. By Theorem 1.17 Cantor's Intersection Theorem,

$$\exists x^* \in \bigcap_{n=1}^{\infty} L_n \ (\subset F)$$

$x^* \notin G_n$ for all $n \in \mathbb{N}$. This implies that $x^* \notin \bigcup_{n=1}^{\infty} G_n$. However this contradicts to the fact that $\{G_n\}_{n \geqq 1}$ is an open cover of F. (x^* is a point of F but not is covered by $\{G_n\}_{n \geqq 1}$.)

So we conclude that the there exists $n \in \mathbb{N}$ s.t

$$F \subset \bigcup_{k=1}^{n} G_k$$

□

28 (Example 8)

STEP 1. For each $x \in F$, we can find $\delta_x > 0$ s.t $B(x, \delta_x) \subset G$ because $x \in F \subset G$. Obviously,

$$F = \bigcup_{x \in F} \{x\} \subset \bigcup_{x \in F} B(x, \delta_x/2).$$

By Theorem 1.21 Heine-Borel Finite Covering Lemma, we can find finite number of points in F, $\{x_1, x_2, \cdots, x_n\}$ and positive numbers $\{\delta_1, \delta_2, \cdots, \delta_n\}$ s.t

$$F \subset \bigcup_{k=1}^{n} B(x_k, \delta_k/2).$$

STEP 2. Let us pick an arbitrary point $x \in F$. Since F is covered by $\{B(x_k, \delta_k/2)\}_{k=1}^{n}$ we can find some i s.t $x \in B(x_i, \delta_i/2)$ Let us pick an arbitrary point $y \in G^c$ and $x \in F$.

$$|x - y| \overset{(*1)}{\geqq} |y - x_i| - |x - x_i| \overset{(*2)}{>} \frac{\delta_i}{2} \geqq \min\{\delta_1/2, \cdots, \delta_n/2\}.$$

- $(*1)$ is obtained by triangular inequality.
- $(*2)$ is because $y \in G^c$ and $B(x_i, \delta_i) \subset G$ so $|y - x_i| \geqq \delta_i$ and $x \in B(x_i, \delta_i/2)$ so $|x - x_i| < \delta_i/2$.

The argument above implies that for $\forall x \in F$ and $\forall z \in \mathbb{R}^d$ with $|z| < \delta^* \overset{\text{def}}{=} \min\{\delta_1/2, \cdots, \delta_n/2\}$, $x + z \in G$. (Conversely, if $y \overset{\text{def}}{=} x + z \in G^c$, $|y - x| = |z| \geqq \min\{\delta_1/2, \cdots, \delta_n/2\}$ by the argument above.) Now the proof is complete.

□

29 (Theorem 1.22)

STEP 1. (E is bounded) For example, $\{B(q, 1)\}_{q \in \mathbb{Q}^d}$ is obviously an open cover of E. We can pick finite number of $\{B(q_i, 1)\}_{i=1}^{k}$ s.t

$$E \subset \bigcup_{i=1}^{k} B(q_i, 1),$$

by assumption. Let $r \overset{\text{def}}{=} \max_{i=1,2\cdots,k}\{|q_i|\} + 1$. Then $\bigcup_{i=1}^{k} B(q_i, 1) \subset B(0, r)$. So E is bounded.

STEP 2. (E is closed) We prove that $E' \subset E$. Let us fix an arbitrary point $y \in E^c$. For each $x \in E$, $x \neq y \Rightarrow |x - y| > 0$. So we can find $\delta_x > 0$ s.t

$$B(x, \delta_x/2) \cap B(y, \delta_x/2) = \emptyset.$$

Since $\{B(x, \delta_x/2)\}_{x \in E}$ is an open cover of E, we can find $\{B(x_i, \delta_{x_i}/2)\}_{i=1}^n$ s.t

$$E \subset \bigcup_{i=1}^n B(x_i, \delta_{x_i}/2).$$

Let $\delta^* \stackrel{\text{def}}{=} \min\{\delta_{x_1}/2, \cdots, \delta_{x_n}/2\}$. Let us choose an arbitrary point $x \in E$. Then we can find $i \in \{1, \cdots, n\}$ s.t $x \in B(x_i, \delta_{x_i}/2)$. Note that

$$|y - x| \overset{(*1)}{\geqq} |y - x_i| - |x_i - x| \overset{*2}{>} \delta_{x_i}/2 \geqq \delta^*$$

- $(*1)$ is obtained by triangular inequality.
- $(*2)$ is because $|y - x_i| > \delta_{x_i}$ ($\because B(x_i, \delta_{x_i}/2) \cap B(y, \delta_{x_i}/2) = \emptyset$) and $x \in B(x_i, \delta_{x_i}/2)$.

This implies that we can not find $\{x_n\}_{n \geqq 1} \subset E$ s.t $x_n \to y$. So y is not limit point of E. In other words, $y \in E^c \Rightarrow y \notin E'$. So

$$E^c \subset (E')^c$$

and this implies that $E' \subset E$.

□

30 (Exercise 1.5.2.9) Please refer to the Example 19 in the next section. Let F be a non-empty closed set and suppose that F' does not contain any isolated point. Then $F' \subset F$ and $F \setminus F' = \emptyset$. So $F = F \setminus F' \cup F' = \emptyset \cup F'$. When $F = F'$, F is called a perfect set. A perfect set is known to be an uncountable set.

□

31 (Exercise 1.5.2.10) Let $\epsilon > 0$ be an arbitrary positive number and let us fix ϵ.

STEP 1. Let $x_i \in F$ be an arbitrary point in F. Since $f_k(x_i) \to +0$ as $k \to \infty$, we can find $N_i \in \mathbb{N}$ s.t $0 \leqq f_{N_i}(x_i) < \frac{\epsilon}{2}$. Moreover $f_{N_i}(x)$ is a continuous function, there exists $\delta_i > 0$ s.t

$$|f_{N_i}(x) - f_{N_i}(x_i)| < \frac{\epsilon}{2}, \ \forall x \in B(x_i, \delta_i).$$

So we have

$$f_{N_i}(x) < \epsilon, \ \forall x \in B(x_i, \delta_i).$$

STEP 2. Note that

$$F \subset \bigcup_{x \in F} B(x, \delta_x),$$

where δ_x is defined in the same way in STEP 1. By Theorem 1.21 Heine-Borel Finite Covering Lemma, we have

$$F \subset \bigcup_{i=1}^n B(x_i, \delta_i).$$

STEP 3. Let $k > \max\{N_1, N_2, \cdots N_n\}$. Note that

$$\sup_{x\in F} f_k(x) \overset{(*1)}{\leqq} \sup_{x\in\bigcup_{i=1}^n B(x_i,\delta_i)} f_k(x) \overset{(*2)}{=} \max_{i=1,2\cdots,n} \sup_{x\in B(x_i,\delta_i)} f_k(x) \overset{(*3)}{\leqq} \max_{i=1,2\cdots,n} \sup_{x\in B(x_i,\delta_i)} f_{N_i}(x) \overset{(*4)}{\leqq} \max_{i=1,2\cdots,n} \epsilon = \epsilon.$$

- $(*1)$ $F \subset \bigcup_{i=1}^n B(x_i, \delta_i)$.
- $(*2)$ See below.
- $(*3)$ $f_k(x)$ is decreasing with respect to n.
- $(*4)$ $f_{N_i}(x) < \epsilon$ for all $x \in B(x_i, \delta_i)$.

This holds for all $n > \max\{N_1, \cdots, N_n\}$. Hence we have $\limsup_{k\to\infty} \sup_{x\in F} f_k(x) \leqq \sup_{x\in F} f_k(x) < \epsilon$, so we conclude that

$$f_k(x) \xrightarrow{u} 0 \text{ on } F.$$

Finally, we present the proof of $(*2)$. First, $\sup_{x\in\bigcup_{i=1}^n B(x_i,\delta_i)} f_k(x) \geqq \sup_{x\in B(x_i,\delta_i)} f_k(x)$, for all $i = 1, 2, \cdots n$. So

$$\sup_{x\in\bigcup_{i=1}^n B(x_i,\delta_i)} f_k(x) \geqq \max_{i=1,\cdots,n} \sup_{x\in B(x_i,\delta_i)} f_k(x).$$

Second, for all $x \in \bigcup_{i=1}^n B(x_i, \delta_i)$, we can find i s.t $x \in B(x_i, \delta_i)$. So $f_k(x) \leqq \sup_{x\in B(x_i,\delta_i)} f_k(x) \leqq \max_{i=1,\cdots,n} \sup_{x\in B(x_i,\delta_i)} f_k(x)$. By taking $\sup_{x\in\bigcup_{i=1}^n B(x_i,\delta_i)}$ of the left hand side, we have

$$\sup_{x\in\bigcup_{i=1}^n B(x_i,\delta_i)} f_k(x) \leqq \max_{i=1,\cdots,n} \sup_{x\in B(x_i,\delta_i)} f_k(x).$$

□

32 (Definition 1.27) $f(x)$ is continuous at $x_0 \in E$ means that

$$\forall\epsilon > 0, \exists\delta > 0 \text{ s.t } \forall x \in B(x_0, \delta) \cap E, |f(x) - f(x_0)| < \epsilon.$$

Equivalently,

$$\lim_{\delta\to+0} \sup_{x\in B(x_0,\delta)\cap E} |f(x) - f(x_0)| = 0,$$

or

$$\lim_{\delta\to+0} \sup_{x\in B(x_0,\delta)\cap E} f(x) = \lim_{\delta\to+0} \inf_{x\in B(x_0,\delta)\cap E} f(x) = f(x_0).$$

(i.e $\lim_{x\to x_0} f(x) = f(x_0)$.) Note that when x_0 is an isolated point of E (i.e $x_0 \in E \setminus E'$), then $f(x)$ is continuous at x_0 by the definition above. When $f(x)$ is continuous at all $x_0 \in E$, we say that $f(x)$ is continuous on E and denote it as $f(x) \in C(E)$. □

33 (Example 9) Suppose that there is not $x \in F$ s.t $f(x) = x$. Since $|f(x) - f(y)| < |x - y|$, $f(x)$ is continuous on F. Let $g(x) \stackrel{\text{def}}{=} |f(x) - x| : F \to [0, \infty)$. $g(x)$ is also continuous on F. And F is bounded and closed. $g(x)$ has a minimum value on F. Suppose that $g(x)$ takes the minimum value at $x_0 \in F$. Since $f(x_0), x_0 \in F$, we have $g(f(x_0)) = |f \circ f(x_0) - f(x_0)| < |f(x_0) - x_0| = g(x_0) > 0$. ($> 0$ holds because $f(x_0) \neq x_0$ by assumption.) Let $x_1 \stackrel{\text{def}}{=} f(x_0) \in F$. Now $g(x_1) < g(x_0)$. (contradiction!!) □

34 (Exercise 1.5.2.11) Let $A \stackrel{\text{def}}{=} \{x \in F \mid f(x) = 0\}$. We show that $A' \subset A$. When $A' = \emptyset$, $A' \subset A$ holds obviously so we may suppose that $A' \neq \emptyset$. Let $a_0 \in A'$, then there exists $\{a_n\} \subset A$ with $a_n \to a_0$. Since $f(a_n) = 0$, we have $\lim_{n\to\infty} f(a_n) = 0$. Since $f(x)$ is continuous, $\lim_{n\to\infty} f(a_n) = f(a_0)$. So $f(a_0) = 0$. And $a_0 \in F' \subset F$. So $a_0 \in A$. $\therefore$ A is a closed set. □

35 (Exercise 1.5.2.12) Let $x_0 \in \bigcup_{n=1}^{\infty} E_n$. We can find $n_0 \in \mathbb{N}$ s.t $x_0 \in E_{n_0}$. We may suppose that $n_0 = 1$ without loss of generality. Since E_1 is an open set, if $\delta > 0$ is sufficiently small, then $B(x_0, \delta) \subset E_1$. So

$$\begin{aligned} & \lim_{\delta\to+0} \sup_{x\in B(x_0,\delta)\cap\bigcup_{n=1}^{\infty} E_n} |f(x) - f(x_0)| \\ = & \lim_{\delta\to+0} \sup_{x\in B(x_0,\delta)\cap E_1} |f(x) - f(x_0)| = 0, \ \because f(x) \in C(E_1) \end{aligned}$$

□

36 (Exercise 1.5.2.13)

(1) Let $f(x) = |x|$. Then $f(x)$ is continuous on E. Since $f(x) = x \in E$ is bounded, E is bounded.

Suppose that E is not closed. So $E' \backslash E$ is not empty. Let $x_0 \in E' \backslash E$. Let $f(x) \stackrel{\text{def}}{=} \frac{1}{|x - x_0|}$. $f(x)$ is continuous and well-defined on E because $x_0 \notin E$. However, $x_0 \in E'$ means that we can find $\{x_n\} \subset E$ s.t $x_n \to x_0$. So $f(x) \to \infty$ as $x_n \to x_0$. This contradicts to the fact that $f(x)$ is bounded. Therefore we conclude that E is closed.

(2) The functions above are non-negative so they have the maximum value means that they are bounded. So we have the same conclusion as the previous question by the same argument.

□

37 (Exercise 1.5.2.14) Let $x_0 \in E$ be an arbitrary point in E. If x_0 is an isolated point ($x_0 \in E \setminus E'$), $f(x)$ is continuous at x_0. So we suppose that $x_0 \in E \cap E'$. Let $\{x_n\}_{n\geqq 1} \subset E$ be an arbitrary sequence with $x_n \to x_0$. Since $K \stackrel{\text{def}}{=} \{x_n\}_{n\geqq 1} \cup \{x_0\}$ is a compact set, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$. Now the proof is complete. □

38 (Definition 1.28) If a set is a countable union of closed sets, then it is called a F_σ set. If a set is a countable intersection of open sets, then it is called G_δ set. (F: closed, G: open, σ: countable union, δ: countable intersection) □

39 (Example 11) A set of continuity of $f(x)$ on G is

$$\{x \in G \mid \omega_f(x) = 0\} = \bigcap_{n=1}^{\infty} \left\{x \in G \mid \omega_f(x) < \frac{1}{n}\right\},$$

where

$$\omega_f(x) \stackrel{\text{def}}{=} \lim_{\delta \to +0} \sup_{x_1, x_2 \in B(x,\delta)} |f(x_1) - f(x_2)|,$$

which is defined in Example 7 of the previous subsection. In Example 7, we have already verified that

$$\{x \in G \mid \omega_f(x) < t\}$$

is an open set for all $t \in \mathbb{R}$ when G is an open set. So the proof is complete. □

40 (Example 12) Let

$$\begin{aligned} A &\stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid f \text{ is continuous at } x\}, \\ B &\stackrel{\text{def}}{=} \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \mathring{E}_k\left(\frac{1}{m}\right), \\ E_k(\epsilon) &\stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^d \mid |f(x) - f_k(x)| \leqq \epsilon\right\}. \end{aligned}$$

We claim that

$$A = B.$$

STEP 1. $(A \subset B)$ Let $x_0 \in A$. We prove that $x_0 \in B$. First, x_0 is a point of continuity of $f(x)$, we have $\forall \epsilon$, there exists $\delta > 0$ s.t

$$|f(x) - f(x_0)| < \epsilon/3, \ \forall x \in B(x_0, \delta).$$

Second, since $f_k(x_0) \to f(x_0)$, there exists sufficiently large $k_0 \in \mathbb{N}$ s.t

$$|f_{k_0}(x_0) - f(x_0)| < \epsilon/3.$$

Third, since $f_{k_0}(x)$ is a continuous function, there exists $\delta' > 0$ s.t

$$|f_{k_0}(x) - f_{k_0}(x_0)| < \epsilon/3, \ \forall x \in B(x_0, \delta').$$

Now let $\delta^* \stackrel{\text{def}}{=} \min(\delta, \delta')$ and we have

$$\begin{aligned} |f(x) - f_{k_0}(x)| &= |f(x) - f(x_0) + f(x_0) - f_{k_0}(x_0) + f_{k_0}(x_0) - f_{k_0}(x)| \\ &\leqq |f(x) - f(x_0)| + |f(x_0) - f_{k_0}(x_0)| + |f_{k_0}(x_0) - f_{k_0}(x)| \\ &\leqq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \forall x \in B(x_0, \delta^*). \end{aligned}$$

This implies that

$$B(x_0, \delta^*) \subset E_{k_0}(\epsilon).$$

Moreover the left hand side is an open set, so we have

$$B(x_0, \delta^*) \subset \mathring{E}_{k_0}(\epsilon).$$

It is easy to see that

$$B(x_0,\delta^*) \subset \bigcup_{k=1}^{\infty} \mathring{E}_k(\epsilon).$$

Now we have

$$x_0 \in \bigcup_{k=1}^{\infty} \mathring{E}_k(\epsilon),\ \forall \epsilon > 0.$$

Therefore

$$x_0 \in \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \mathring{E}_k\left(\frac{1}{m}\right)$$

STEP 2. $(B \subset A)$ Let $x_0 \in B$. We prove that $x_0 \in A$. Our goal is to prove that $\forall \epsilon > 0$, $\exists \delta > 0$ s.t

$$|f(x) - f(x_0)| < \epsilon,\ \forall x \in B(x_0,\delta).$$

First we take $m_0 \in \mathbb{N}$ s.t $\frac{1}{m_0} < \frac{\epsilon}{3}$. Note that

$$x_0 \in \bigcup_{k=1}^{\infty} \mathring{E}_k\left(\frac{1}{m_0}\right).$$

We can find k_0 s.t

$$x_0 \in \mathring{E}_{k_0}\left(\frac{1}{m_0}\right).$$

Since the right hand side is a set of interior points, we can find $\delta_0 > 0$ s.t

$$B(x_0,\delta_0) \in \mathring{E}_{k_0}\left(\frac{1}{m_0}\right).$$

Also note that

$$\mathring{E}_{k_0}\left(\frac{1}{m_0}\right) \subset E_{k_0}\left(\frac{1}{m_0}\right).$$

So we find out that

$$|f_{k_0}(x) - f(x)| \leqq \frac{1}{m_0} < \frac{\epsilon}{3},\ \forall x \in B(x_0,\delta_0).$$

Note that $x_0 \in B(x_0,\delta_0)$, so we have

$$|f_{k_0}(x_0) - f(x_0)| \leqq \frac{1}{m_0} < \frac{\epsilon}{3}.$$

Let us recall that $f_{k_0}(x)$ is a continuous function on $\mathbb{R}^d$. So $f_{k_0}(x)$ is continuous at x_0. This implies that there exists $\delta_1 > 0$ s.t

$$|f_{k_0}(x) - f_{k_0}(x_0)| < \frac{\epsilon}{3},\ \forall x \in B(x_0,\delta_1).$$

Finally let $\delta \overset{\text{def}}{=} \min\{\delta_0,\delta_1\}$. We have

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_{k_0}(x) + f_{k_0}(x) - f_{k_0}(x_0) + f_{k_0}(x_0) - f(x_0)| \\ &\leqq |f(x) - f_{k_0}(x)| + |f_{k_0}(x) - f_{k_0}(x_0)| + |f_{k_0}(x_0) - f(x_0)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,\ \forall x \in B(x,\delta) \end{aligned}$$

STEP 3. (A is a G_δ set) $A = B$ and B is obviously a G_δ set. Now the proof is complete.

□

41 (Definition 1.29, 1.30, 1.31)

(1) Let $\mathscr{A}$ be a collection of point sets. ($\forall A \in \mathscr{A}$, A is a point set.) If $\mathscr{A}$ satisfies the following conditions, we say that $\mathscr{A}$ is a σ-algebra.

- $\emptyset \in \mathscr{A}$
- if $\forall A \in \mathscr{A}$, then $A^c \in \mathscr{A}$
- if $\{A_n\}_{n=1}^{\infty} \subset \mathscr{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$

(2) Let Σ be a collection of point sets. And let $\{\mathscr{A}_i\}_{i\in I}$ be a collection of σ-algebras with $\Sigma \subset \mathscr{A}_i, \forall i \in I$. Then $\mathscr{A} \overset{\text{def}}{=} \bigcap_{i\in I} \mathscr{A}_i$ is also a σ-algebra. (the proof is easy.) We also denote $\mathscr{A}$ as $\sigma[\Sigma]$. This is called a $\sigma-$algebra generated from Σ. We can also say that this is the smallest $\sigma-$algebra that contains Σ.

(3) Let $\mathscr{O}^d$ be a collection of all open set on $\mathbb{R}^d$. Then $\sigma\left[\mathscr{O}^d\right]$ is called Borel algebra, or Borel sigma algebra. Each element in $\sigma\left[\mathscr{O}^d\right]$ is called a Borel set. We often denote it as $\mathscr{B} \overset{\text{def}}{=} \sigma\left[\mathscr{O}^d\right]$.

□

42 (Exercise 1) We claim that

$$A \overset{\text{def}}{=} \{x \in [a,b] \mid f(x) < t\} = B \overset{\text{def}}{=} \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\left\{x \in [a,b] \mid f_m(x) \leqq t - \frac{1}{k}\right\}.$$

STEP 1. ($A \subset B$) Let $x_0 \in A$. Since $f(x_0) < t$, there exists sufficiently large $k_0 \in \mathbb{N}$ s.t

$$f(x_0) < t - \frac{1}{k_0}$$

Since $f_n(x_0) \to f(x_0)$, there exists $n_0 \in N$ s.t $\forall m \geqq n_0$,

$$f_m(x_0) < t - \frac{1}{k_0}.$$

This implies that

$$x_0 \in \bigcap_{m=n_0}^{\infty}\left\{x \in [a,b] \mid f_m(x) \leqq t - \frac{1}{k_0}\right\}$$

and note that

$$\begin{aligned}\bigcap_{m=n_0}^{\infty}\left\{x \in [a,b] \mid f_m(x) \leqq t - \frac{1}{k_0}\right\} &\subset \bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\left\{x \in [a,b] \mid f_m(x) \leqq t - \frac{1}{k_0}\right\} \\ &\subset \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\left\{x \in [a,b] \mid f_m(x) \leqq t - \frac{1}{k}\right\}\end{aligned}$$

So $x_0 \in B$.

STEP 2. $(B \subset A)$ Let $x_0 \in B$. There exists $k_0, n_0 \in \mathbb{N}$ s.t

$$f_m(x_0) \leqq t - \frac{1}{k_0}, \forall m \geqq n_0.$$

This implies that

$$f(x_0) = \limsup_{n \to \infty} f_n(x_0) \leqq t - \frac{1}{k_0} < t.$$

So $x_0 \in A$.

STEP 3. Since every $f_n(x)$ is a continuous function,

$$\bigcap_{m=n}^{\infty} \left\{ x \in [a, b] \mid f_m(x) \leqq t - \frac{1}{k} \right\}$$

is a closed set for each $n \in \mathbb{N}, k \in \mathbb{N}$. (See Theorem 1.16 and Example 2 in the previous section.) So B is a F_σ set.

□

43 (Exercise 2) We show that

$$A = \left\{ x \in F \mid \liminf_{n \to \infty} f_n(x) > a \right\}, \ (a \in \mathbb{R})$$

is a F_σ set. (Then the rest proof is easy.) To prove the above statement, we claim that $A = B$ where

$$B = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \left\{ x \in F \mid f_m(x) \geqq a + \frac{1}{k} \right\}.$$

(It is easy to prove that B is a F_σ set.)

STEP 1. $(A \subset B)$ First suppose that $x_0 \in A$. Then

$$\liminf_{n \to \infty} f_n(x_0) > a.$$

This implies that we can find $k_0 \in \mathbb{N}$ s.t

$$\liminf_{n \to \infty} f_n(x_0) > a + \frac{1}{k_0}$$

Let us define

$$g_n(x) \stackrel{\text{def}}{=} \inf_{m \geqq n} f_m(x).$$

Note that

$$\liminf_{n \to \infty} f_n(x_0) = \lim_{n \to \infty} g_n(x_0) > a + \frac{1}{k_0}$$

Since $g_n(x_0)$ is monotone increasing with respect to n, we can find n_0 s.t

$$g_{n_0}(x_0) > a + \frac{1}{k_0}.$$

So

$$f_m(x_0) > a + \frac{1}{k_0},\ \forall m \geqq n_0.$$

This implies that

$$\begin{aligned}
x_0 \in \bigcap_{m=n_0}^{\infty} \left\{ x \in F \mid f_m(x) > a + \frac{1}{k_0} \right\} \quad &\subset \quad \bigcap_{m=n_0}^{\infty} \left\{ x \in F \mid f_m(x) \geqq a + \frac{1}{k_0} \right\} \\
&\subset \quad \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \left\{ x \in F \mid f_m(x) \geqq a + \frac{1}{k_0} \right\} \\
&\subset \quad \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \left\{ x \in F \mid f_m(x) \geqq a + \frac{1}{k} \right\} \\
&= \quad B
\end{aligned}$$

STEP 2. $(B \subset A)$ Let $x_0 \in B$. There exists $k_0, n_0 \in \mathbb{N}$ s.t

$$f_m(x_0) \geqq a + \frac{1}{k_0},\ \forall m \geqq n_0.$$

This implies that

$$\inf_{m \geqq n_0} f_m(x_0) \geqq a + \frac{1}{k_0},$$

and hence

$$\liminf_{n\to\infty} f_n(x_0) \geqq \inf_{m \geqq n_0} f_m(x_0) \geqq a + \frac{1}{k_0} > a.$$

This implies that $x_0 \in A$.

STEP 3. (Proof of the rest part)

$$\begin{aligned}
& \{x \in F \mid f_n(x) \text{ converges at x}\} \\
= \quad & \left\{ x \in F \mid \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x) \right\} \\
\overset{*1}{=} \quad & \left\{ x \in F \mid \limsup_{n\to\infty} f_n(x) > \liminf_{n\to\infty} f_n(x) \right\}^c \\
\overset{*2}{=} \quad & \left(\bigcup_{r\in\mathbb{Q}} \bigcup_{n=1}^{\infty} \left\{ x \in F \mid \limsup_{n\to\infty} f_n(x) \geqq r + \frac{1}{2n} \text{ and } r - \frac{1}{2n} \geqq \liminf_{n\to\infty} f_n(x) \right\} \right)^c \\
= \quad & \bigcap_{r\in\mathbb{Q}} \bigcap_{n=1}^{\infty} \left\{ x \in F \mid \limsup_{n\to\infty} f_n(x) < r + \frac{1}{2n} \text{ or } r - \frac{1}{2n} < \liminf_{n\to\infty} f_n(x) \right\} \\
= \quad & \bigcap_{r\in\mathbb{Q}} \bigcap_{n=1}^{\infty} \left\{ x \in F \mid \limsup_{n\to\infty} f_n(x) < r + \frac{1}{2n} \right\} \cup \left\{ x \in F \mid r - \frac{1}{2n} < \liminf_{n\to\infty} f_n(x) \right\}
\end{aligned}$$

- $(*1)$ Note that $\limsup \geqq \liminf$ alway holds.
- $(*2)$ This is because if $a, b \in \mathbb{R}, a < b$ holds, then we can find a $r \in \mathbb{Q}$ (the set of rational numbers is dense in $\mathbb{R}$.) and sufficiently large $n \in \mathbb{N}$ s.t $[r-1/2n, r+1/2n] \subset [a, b]$. (The converse also holds obviously.)

Finally,

$$\left\{x \in F \mid r - \frac{1}{2n} < \liminf_{n\to\infty} f_n(x)\right\}$$

is a F_σ set by the previous result. And also note that

$$\left\{x \in F \mid \limsup_{n\to\infty} f_n(x) < r + \frac{1}{2n}\right\} = \left\{x \in F \mid -r - \frac{1}{2n} < \liminf_{n\to\infty}(-f_n(x))\right\}.$$

A union of two F_σ sets is also a F_σ set. So we conclude that the set above is a countable intersection of F_σ sets, which is called a $F_{\sigma,\delta}$ set.

□

44 (Exercise 3) The proof is somewhat similar to that of Example 11. Let

$$\tilde{\omega}(x) \stackrel{\text{def}}{=} \lim_{\delta\to+0} \sup_{x_1,x_2\in B(x,\delta)\setminus\{x\}} |f(x_1) - f(x_2)|.$$

Note that $\lim_{x\to x_0} f(x)$ exists if and only if

$$\tilde{\omega}(x_0) = 0.$$

Since

$$\left\{x \in \mathbb{R} \mid \lim_{y\to x} f(y) \text{ exists}\right\} = \bigcap_{n=1}^{\infty} \left\{x \in \mathbb{R} \mid \tilde{\omega}(x) < \frac{1}{n}\right\},$$

it is enough for us to show that

$$\{x \in \mathbb{R} \mid \tilde{\omega}(x) < t\} \text{ is open }, \forall t > 0.$$

Suppose that $x_0 \in \{x \in \mathbb{R} \mid \tilde{\omega}(x) < t\}$ (We assume that $t > 0$ is now fixed.). Since

$$\sup_{x_1,x_2\in B(x,\delta)\setminus\{x\}} |f(x_1) - f(x_2)|$$

is monotone decreasing with respect to $\delta > 0$, if $\tilde{\omega}(x_0) < t$, then we can find $\delta_0 > 0$ s.t

$$\sup_{x_1,x_2\in B(x_0,\delta_0)\setminus\{x_0\}} |f(x_1) - f(x_2)| < t$$

We prove that

$$B(x_0, \delta_0) \subset \{x \in \mathbb{R} \mid \tilde{\omega}(x) < t\},$$

and then the proof is complete. Let us pick an arbitrary point $x^* \in B(x_0, \delta_0)$.

case 1. $(x^* = x_0)$ $x^* = x_0 \in \omega\{x \in \mathbb{R} \mid \tilde{\omega}(x) < t\}$ by assumption.

case 2. $(x^* \neq x_0)$ We can find sufficiently small $\delta^* > 0$ s.t $x_0 \notin B(x^*, \delta^*)$. Note that

$$\begin{aligned}
\tilde{\omega}(x^*) &\stackrel{\text{def}}{=} \lim_{\delta\to+0} \sup_{x_1,x_2\in B(x^*,\delta)\setminus\{x^*\}} |f(x_1) - f(x_2)| \\
&\stackrel{*}{\leqq} \sup_{x_1,x_2\in B(x^*,\delta^*)\setminus\{x^*\}} |f(x_1) - f(x_2)| \\
&\leqq \sup_{x_1,x_2\in B(x^*,\delta^*)} |f(x_1) - f(x_2)| \\
&\leqq \sup_{x_1,x_2\in B(x_0,\delta_0)\setminus\{x_0\}} |f(x_1) - f(x_2)| < t,
\end{aligned}$$

because

$$B(x^*,\delta^*)\setminus\{x^*\}\subset B(x^*,\delta^*)=B(x^*,\delta^*)\setminus\{x_0\}\subset B(x_0,\delta_0)\setminus\{x_0\}.$$

- $(*)$ Note that $\sup_{x_1,x_2\in B(x^*,\delta)\setminus\{x^*\}}|f(x_1)-f(x_2)|$ decreases as $\delta\to+0$.

Now the proof is complete.

□

45 (Theorem 1.23 Baire) We suppose that $E=\bigcup_{k=1}^{\infty}F_k$ has an interior point and derive a contradiction. Let us pick an interior point $x_0\in E$. There exists $\delta_0>0$ s.t

$$\overline{B}(x_0,\delta_0)\subset E.$$

This is possible because we can pick $\delta_0^*>0$ s.t $B(x_0,\delta_0^*)\subset E$ and pick $\delta_0\in(0,\delta_0^*)$ again, then we have $\overline{B}(x_0,\delta_0)\subset B(x_0,\delta_0^*)\subset E$.

STEP 1. (pick x_1) We pick $x_1\in B(x_0,\delta_0)\setminus F_1$. $B(x_0,\delta_0)\setminus F_1$ is not empty because $B(x_0,\delta_0)\subset F_1$ can not occur. Otherwise, x_0 is an interior point of F_1 and this contradicts to the assumption that F_1 has no interior point.

Since $B(x_0,\delta_0)$ is an open set, we can find $\delta_1>0$ s.t

$$\overline{B}(x_1,\delta_1)\subset B(x_0,\delta_0).$$

Moreover, by taking sufficiently small $\delta_1>0$, we can satisfy

$$\overline{B}(x_1,\delta_1)\cap F_1=\emptyset,$$

at the same time. Otherwise, for all small $\delta_1>0$, $\overline{B}(x_1,\delta_1)\cap F_1\neq\emptyset$ implies that we can find a sequence $\{x_{1,n}\}\subset F_1$ s.t $x_{1,n}\to x_1$. So $x_1\in F_1'\subset F_1$ and this contradicts to the fact that $x_1\in B(x_0,\delta_0)\setminus F_1$.

STEP 2. (pick x_2) Let us repeat a similar argument. Let us pick $x_2\in B(x_1,\delta_1)\setminus F_2$. $B(x_1,\delta_1)\setminus F_2$ is not an empty set because $B(x_1,\delta_1)\not\subset F_2$ can not happen because F_2 has no interior point. We can find small $\delta_2>0$ s.t

$$\overline{B}(x_2,\delta_2)\subset B(x_1,\delta_1),\ \text{and}\ \overline{B}(x_2,\delta_2)\cap F_2=\emptyset,$$

because $B(x_1,\delta_1)$ is an open set and if the second statement does not hold, we can find $\{x_{2,n}\}\subset F_2$ s.t $x_{2,n}\to x_2\in F_2'\subset F_2$ and this contradicts to the fact that $x_2\notin F_2$.

STEP 3. (pick x_k) Simiarly, we can find x_k and δ_k s.t

$$\overline{B}(x_k,\delta_k)\subset B(x_{k-1},\delta_{k-1})\subset E,\ \text{and}\ \overline{B}(x_k,\delta_k)\cap F_k=\emptyset.$$

Without loss of generality, we may suppose that

$$0<\delta_k<\frac{1}{k},$$

because the conditions above hold as long as δ_k is small enough. We claim that $\{x_k\}\subset\mathbb{R}$ is a Cauchy sequence. Let us consider $\ell\geqq k$. Then $x_\ell\in B(x_k,\delta_k)$. So $|x_\ell-x_k|\leqq\frac{1}{k}$ and hence

$$\lim_{k,\ell\to\infty}|x_k-x_\ell|=0.$$

By completeness of $\mathbb{R}$, x_k converges to $x\in\mathbb{R}$.

STEP 4. (derive contradiction) Let $\ell \geqq k \geqq 1$. By triangular inequality and since $x_\ell \in B(x_k, \delta_k)$, we have

$$\begin{aligned} |x - x_k| &\leqq |x - x_\ell| + |x_k - x_\ell| \\ &\leqq |x - x_\ell| + \delta_k \end{aligned}$$

This holds for all $\ell \geqq k$. By taking $\ell \to \infty$, $|x - x_k| \leqq \delta_k$. So $x \in \overline{B}(x_k, \delta_k) \subset B(x_0, \delta_0) \subset E$. However, since $x \in \overline{B}(x_k, \delta_k)$, $x \notin F_k$ for all $k \geqq 1$ (because $\overline{B}(x_k, \delta_k) \cap F_k = \emptyset$) , and hence $x \notin \bigcup_{k=1}^{\infty} F_k = E$. This contradicts to the fact that $x \in E$.

□

46 (Example 13) When $A \subset \mathbb{R}$ and $\overline{A} \overset{\text{def}}{=} A \cup A' = \mathbb{R}$, we say that A is dense in $\mathbb{R}$.

STEP 1. First we prove that if A is dense in $\mathbb{R}$, then $A^c \overset{\text{def}}{=} \mathbb{R}^d \setminus A$ has no interior point. We consider its contraposition. If A^c has an interior point, then A is not dense. This is obvious because there exists $x \in A^c$ and $\delta > 0$ s.t $B(x, \delta) \subset A^c$. Then we can not take $\{a_n\} \subset A$ s.t $a_n \to x$ because when n is sufficiently large, $|a_n - x| < \delta$, but then $a_n \in B(x, \delta) \subset A^c$ and this contradicts to the fact that $\{a_n\} \subset A$.

STEP 2. Suppose that $\mathbb{Q}$ is a G_δ set. So there exists a countable number of open sets $\{G_k\}_{k \geqq 1}$ s.t

$$\mathbb{Q} = \bigcap_{k=1}^{\infty} G_k.$$

From this equation, we find out that $\mathbb{Q} \subset G_k$ for all $k \geqq 1$. Since $\mathbb{Q}$ is a dense set, G_k is also dense in $\mathbb{R}$. Let $F_k \overset{\text{def}}{=} \mathbb{R} \setminus G_k$ (By STEP 1, F_k has no interior point.) and let $\bigcup_{n=1}^{\infty} \{q_n\} \overset{\text{def}}{=} \mathbb{Q}$. (For each $n \in \mathbb{N}$, a single point $\{q_n\}$ is also a closed set with no interior point.) Note that

$$\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \bigcup_{k=1}^{\infty} F_k \cup \bigcup_{k=1}^{\infty} \{q_k\},$$

so $\mathbb{R}$ is a countable union of closed sets with no interior point. By Theorem 1.23 (Baire), $\mathbb{R}$ has no interior point. (contradiction!!) Now the proof is complete.

□

47 (Definition 1.32)

(1) Suppose $A \subset \mathbb{R}^d$ and $\overline{A} \overset{\text{def}}{=} A \cup A' = \mathbb{R}^d$. Then we say that A is dense in $\mathbb{R}^d$. If $A \subset E$ and $\overline{A} = E$, then we say that A is dense in E.

(2) Let $E \subset \mathbb{R}^d$. Suppose that $\mathring{\overline{E}} = \emptyset$ ($\overline{E}$ has no interior point). Then we say that E is a nowhere dense set.

(3) If E is a countable union of nowhere dense sets, then we say that E is a meagre set or a set of first category. If E is not a meagre set, we say that E is a set of second category.

□

48 (Example 14)

STEP 1. Let $A \subset \mathbb{R}^d$. Suppose that $A^c \stackrel{\text{def}}{=} \mathbb{R}^d \setminus A$ has no interior point, then A is dense in $\mathbb{R}^d$. (Equivalently, A is not dense in $\mathbb{R}^d$, then A^c has at least one interior point.) Let us fix an arbitrary point $x \in A^c$. Since A^c has no interior point, $\forall \delta > 0$, $B(x, \delta) \not\subset A^c$. This implies that $B(x, \delta) \setminus A^c = B(x, \delta) \cap A \neq \emptyset$. By taking small $\delta > 0$, we can find a sequence $\{x_n\} \subset A$ s.t $x_n \to x$ ($x_i \neq x_j$ if $i \neq j$). In other words, $A^c \subset A'$. So we have $\mathbb{R}^d = A \cup A^c \subset A' \cup A = \overline{A}$. (Now we find out that A is dense if and only if A^c has no interior point. Also see Example 13.)

STEP 2. Let $F_k \stackrel{\text{def}}{=} \mathbb{R}^d \setminus G_k$. Suppose that $\bigcap_{k=1}^\infty G_k$ is not dense. Then $(\bigcap_{k=1}^\infty G_k)^c = \bigcup_{k=1}^\infty F_k$ has at least one interior point.

Since every G_k is dense, F_k has no interior point. (See Example 13.) By Theorem 1.32 (Baire), $\bigcup_{k=1}^\infty F_k$ has no interior point. This contradicts to the fact stated above. Now the proof is complete.

□

49 (Example 15) In Example 12, we have already shown that

$$\{x \in \mathbb{R}^d \mid f \text{ is continuous at } x\} = \bigcap_{m=1}^\infty \bigcup_{k=1}^\infty \mathring{E}_k(1/m),$$

where

$$E_k(\epsilon) \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^d \mid |f_k(x) - f(x)| \leqq \epsilon\right\}.$$

Let $G(\epsilon) \stackrel{\text{def}}{=} \bigcup_{k=1}^\infty \mathring{E}_k(\epsilon)$, and we show that $G(\epsilon)^c$ is a meagre set. Then $\bigcup_{m=1}^\infty G(1/m)^c$ is a meagre set.

STEP 1. Let us fix $\epsilon > 0$, which is an arbitrary positive numbers. Let

$$F_k(\epsilon) = \bigcap_{\ell=1}^\infty \{x \in \mathbb{R}^d \mid |f_k(x) - f_{k+\ell}(x)| \leqq \epsilon\}.$$

Note that $F_k(\epsilon)$ is closed because $f_k(x), f_{k+\ell}(x)$ are closed and an intersection of closed sets is also closed. We claim that

$$\mathbb{R}^d = \bigcup_{k=1}^\infty F_k(\epsilon).$$

Let us pick arbitrary point $x \in \mathbb{R}^d$ and fix x for now. Note that

$$\lim_{k,j\to\infty} |f_k(x) - f_j(x)| \leqq \lim_{k,j\to\infty} |f_k(x) - f(x)| + |f_j(x) - f(x)| = 0,$$

because $f_k(x) \to f(x)$. This implies that there exists sufficiently large $k_0 \in \mathbb{N}$ s.t

$$|f_k(x) - f_j(x)| \leqq \epsilon, \ \forall k, j \geqq k_0.$$

This also implies that
$$|f_{k_0}(x) - f_{k_0+\ell}(x)| \leqq \epsilon,\ \forall \ell \geqq 1.$$
So we have
$$\begin{aligned} x &\in \bigcap_{\ell=1}^{\infty} \{x \in \mathbb{R}^d \mid |f_{k_0}(x) - f_{k_0+\ell}(x)| \leqq \epsilon\} \\ &\subset \bigcup_{k=1}^{\infty} \bigcap_{\ell=1}^{\infty} \{x \in \mathbb{R}^d \mid |f_k(x) - f_{k+\ell}(x)| \leqq \epsilon\} \\ &= \bigcup_{k=1}^{\infty} F_k(\epsilon) \end{aligned}$$

STEP 2. We claim that
$$F_k(\epsilon) \subset E_k(\epsilon).$$
Let $x \in F_k(\epsilon)$, then we have.
$$\begin{aligned} |f_k(x) - f(x)| &\leqq |f_k(x) - f_{k+\ell}(x)| + |f_{k+\ell}(x) - f(x)| \\ &\leqq \epsilon + |f_{k+\ell}(x) - f(x)| \to \epsilon \text{ as } \ell \to \infty. \end{aligned}$$
So $x \in E_k(\epsilon)$. This implies that $F_k(\epsilon) \subset E_k(\epsilon)$.

STEP 3. Note that
$$\mathring{F}_k(\epsilon) \subset F_k(\epsilon) \subset E_k(\epsilon) \subset G(\epsilon),$$
so we have
$$\bigcup_{k=1}^{\infty} \mathring{F}_k(\epsilon) \subset G(\epsilon),$$
therefore,
$$\begin{aligned} G(\epsilon)^c = \mathbb{R}^d \setminus G(\epsilon) &\subset \mathbb{R}^d \setminus \bigcup_{k=1}^{\infty} \mathring{F}_k(\epsilon) \\ &= \bigcup_{k=1}^{\infty} F_k(\epsilon) \setminus \bigcup_{k=1}^{\infty} \mathring{F}_k(\epsilon) \\ &\subset \bigcup_{k=1}^{\infty} F_k(\epsilon) \setminus \mathring{F}_k(\epsilon) \\ &= \bigcup_{k=1}^{\infty} \partial F_k(\epsilon) \end{aligned}$$

We show the following two facts, and then the proof is complete.

- ∂F is a nowhere dense set when F is a closed set.
- A subset of a meagre set is also a meagre set.

STEP 4. (∂F is a nowhere dense set if F is closed) We show that $\overline{\partial F}$ has no interior point. Note that $\partial F = \overline{F} \setminus \mathring{F}$ is a closed set. So $\overline{\partial F} = \partial F$. We show ∂F has no interior point. Suppose ∂F has an interior point. Then there exists B a non-empty open set s.t $B \subset \partial F \subset F$. From this, we find out that

$$B \subset \partial F, \text{ and } B \subset \mathring{F}.$$

(B is an open set and B is a subset of F. We can say that every point of B is an interior point of F. So $B \subset \mathring{F}$.) So

$$B \subset \partial F \cap \mathring{F}.$$

But the right hand side is an empty set. (contradiction!!) So we concldue ∂F has no interior point.

STEP 5. (A subset of a meagre set is also a meagre set.) Suppose that A is a meagre set. Then there exist nowhere dense sets $\{E_k\}_{k \geqq 1}$ s.t $A = \bigcup_{k=1}^{\infty} E_k$. Let $B \subset A$. Then $B = \bigcup_{k=1}^{\infty} E_k \cap B$. Since $E_k \cap B \subset E_k$, $E_k \cap B$ is also a nowhere dense set.

□

50 (Cantor Set: Definition and Properties) Let us define $\{C_n\}_{n \geqq 1}$ in the following way.

- $C_0 \stackrel{\text{def}}{=} [0, 1]$.
- $C_1 \stackrel{\text{def}}{=} [0, 1/3] \cup [2/3, 1]$.
- $C_2 \stackrel{\text{def}}{=} [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$.
- $C_n \stackrel{\text{def}}{=} \bigcup_{k=1}^{2^n} I_{n,k}$.

The rule is easy. C_n consists of 2^n closed intervals. We divide each closed interval into three peaces, and then remove the one in the middle. For example, if $n = 1$, we divide $C_0 = [0, 1]$ into $[0, 1/3] \cup [1/3, 2/3] \cup [2/3, 1]$ and remove $[1/3, 2/3]$. Note that $C_{n+1} \subset C_n$. Finally

$$C \stackrel{\text{def}}{=} \bigcap_{n=1}^{\infty} C_n.$$

C is called a Cantor set defined on $[0, 1]$.

(1) Obviously, $C \subset [0, 1]$. So C is bounded. Since C_n is closed for all $n \in \mathbb{N}$, their countable intersection $C = \bigcap_{n=1}^{\infty} C_n$ is also closed. (Theorem 1.16)

(2) Since C is closed, $C' \subset C$. We show that $C \subset C'$. Let us pick an arbitrary point $x \in C$. Then $x \in C_n$ for all $n \in \mathbb{N}$ and there exists k ($1 \leqq k \leqq 2^n$) s.t $x \in I_{n,k}$. Let us pay attention to the fact that the edge points of $I_{n,k}$ are contained in C. Therefore at least one of the edge point of $I_{n,k}$ is not x. Let $x_n \in I_{n,k}$ be the edge point of $I_{n,k}$ with $x_n \neq x$.

Now we have a sequence of $\{x_n\}_{n \geqq 1}$ with $0 < |x_n - x| \leqq \frac{1}{3^n}$. (because the length of interval is $\frac{1}{3^n}$.) From this inequality, we can assume that x_n are different each other, because we can find a subsequence so that $\{x_n\}$ are different from each other. (Even if $x_n = x_{n+1}\cdots$, we can take larger n^* such that $|x_{n^*} - x| \leqq \frac{1}{3^{n^*}} < |x_n - x|$.) Note that $\{x_n\} \subset C$ s.t $x_n \to x$. Therefore $x \in C'$. So we conclude that $C \subset C'$.

(3) Let $x \in C$ and let $\delta > 0$ be an arbitrary small positive number. Let us take sufficiently large n such that $\frac{1}{3^n} < \delta$. $x \in C_n$ for all $n \geqq 1$ and we can find k $(1 \leqq k \leqq 2^n)$ s.t $x \in I_{n,k}$. When constructing C_{n+1}, the middle part of $I_{n,k}$ will be removed and the removed part is not contained in C. This implies that $B(x, \frac{1}{3^n}) \subset B(x,\delta)$ contains points which are not in C. So x is not an interior point of C. We conclude that C has no interior point.

□

51 (Example 17 Cantor function)

(1) Let us construct a sequence of continuous functions $\{\Phi_n(x)\}$ defined on $[0,1]$ shown in the figures below. (See the figures.) Let us recall how to construct a Cantor set.

STEP 1. In constructing C_1, we remove $(1/3, 2/3)$. So $\Phi_1(x) = 1/2$ for $x \in (1/3, 2/3)$. And we connect $(0,0)$ with $(1/3, 1/2)$ and $(2/3, 1/2)$ with $(1,1)$ so that $\Phi_1(x)$ becomes a continuous function on $[0,1]$.

STEP 2. Since $(1/3, 2/3)$ is already removed, we use the same definition on the removed part. (i.e $\Phi_2(x) = 1/2$ for $x \in (1/3, 2/3)$.) And we update the definition on other parts. In constructing C_2, we remove $(1/9, 2/9)$ and $(7/9, 8/9)$. So $\Phi_2(x) = 1/4$ for $x \in (1/9, 2/9)$ and $\Phi_2(x) = 3/4$ for $x \in (7/9, 8/9)$. And we connect the dots again so that the $\Phi_2(x)$ becomes a continuous function on $[0,1]$.

STEP 3. We continue the similar procedure and obtain $\{\Phi_n(x)\}_{n\geqq 1} \subset C([0,1])$. Finally, $\Phi(x) \overset{\text{def}}{=} \lim_{n\to\infty} \Phi_n(x)$. (We will prove why this limit exists and that $\Phi(x)$ is continuous.) $\Phi(x)$ is called a Cantor function.

(2) We prove that $\Phi_n(x) \overset{u}{\to} \Phi(x)$ (converge uniformly) on $[0,1]$. It is easy to see that

$$|\Phi_n(x) - \Phi_{n-1}(x)| \leqq \frac{1}{2^n}.$$

Therefore $\sum_{n=1}^{\infty} |\Phi_n(x) - \Phi_{n-1}(x)| < \infty$. Absolute convergence implies convergence. (i.e $\sum_{n=1}^{\infty} |a_n| < \infty$ implies $\sum_{n=1}^{\infty} a_n$ converges.) So

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\Phi_k(x) - \Phi_{k-1}(x)) + \Phi_0(x)$$

converges. So we conclude that

$$\lim_{n\to\infty} \Phi_n(x) \text{ converges.}$$

Let $\Phi(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} \Phi_n(x)$. Note that

$$\begin{aligned}
|\Phi_n(x) - \Phi(x)| &= \lim_{m\to\infty} |\Phi_n(x) - \Phi_m(x)| \\
&= \lim_{m\to\infty} |\Phi_n(x) - \Phi_{n+1}(x) + \Phi_{n+1}(x) - \cdots + \Phi_m(x)| \\
&\leqq \lim_{m\to\infty} \sum_{k=n+1}^{m} |\Phi_k(x) - \Phi_{k-1}(x)| \\
&\leqq \lim_{m\to\infty} \sum_{k=n+1}^{m} \frac{1}{2^k} \\
&= \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}
\end{aligned}$$

This implies that

$$\sup_{x\in[0,1]} |\Phi_n(x) - \Phi(x)| \leqq \frac{1}{2^n} \to 0 \text{ as } n \to \infty.$$

Since $\Phi_n(x) \xrightarrow{u} \Phi(x)$ on $[0,1]$ and $\{\Phi_n(x)\}_{n\geqq 1} \subset C([0,1])$, $\Phi(x) \in C([0,1])$. (Recall that if a sequence of continuous functions uniformly converges, then the limit is also a continuous function.)

□

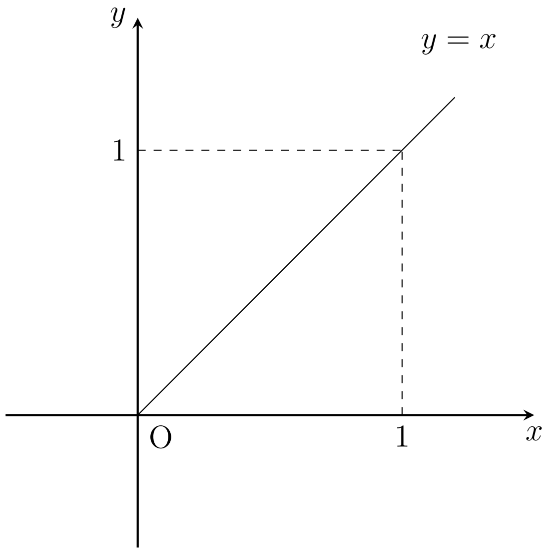

Figure 1.1: $\Phi_0(x)$

52 (Example 18)

STEP 1. (⇒) Suppose that E is a perfect set. $E = E'$ implies that $E' \subset E$ so E is a closed set. Therefore $E^c \subset \mathbb{R}$ is an open set. By Theorem 1.19, we there exists

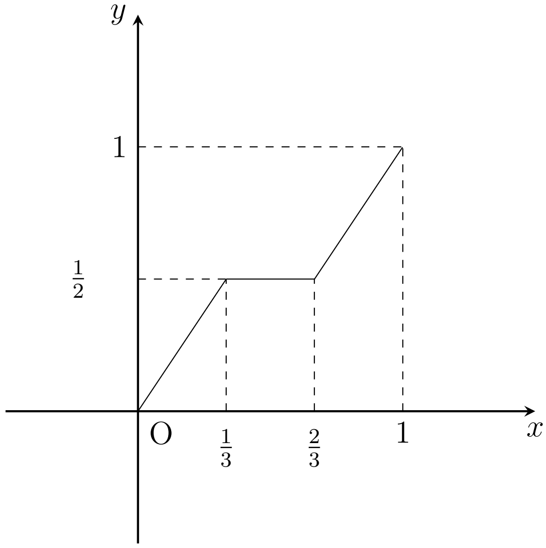

Figure 1.2: $\Phi_1(x)$

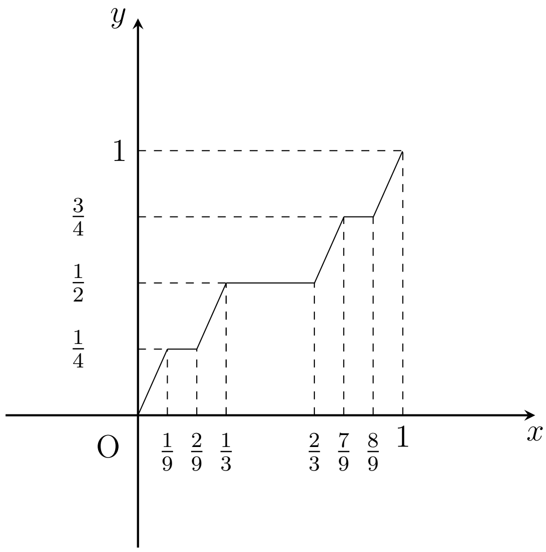

Figure 1.3: $\Phi_2(x)$

countable number of disjoint open intervals s.t

$$E^c = \bigcup_{n=1}^{\infty} (a_n, b_n).$$

We show that $\{(a_n, b_n)\}$ have no common edge point. We suppose that $\{(a_n, b_n)\}$ have a common edge point. Assume that $(a_1, b_1), (a_2, b_2)$ have the common edge point $b_1 = a_2$. Let $x^* \stackrel{\text{def}}{=} b_1 = a_2$. Then x^* is not contained in E^c so $x^* \in E$. x^* is an isolated point of E because for any small $\delta > 0$, $B(x^*, \delta) \cap E = \{x^*\}$. However, a perfect set does not have an isolated point because $E \setminus E' = \emptyset$ $(E = E')$. (Let us recall that $E \setminus E'$ is a set of isolated point of E.) Now the proof of $\Rightarrow$ is complete.

STEP 2. ($\Leftarrow$) Suppose that

$$E^c \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty}(a_n, b_n),$$

where $\{(a_n, b_n)\}$ have no common edge point. We show that E is a perfect set. From the equality above, E^c is an open set. (A countable union of open sets is also an open set.) Therefore E is a closed set, and we have $E' \subset E$. It is enough for us to prove that $E \subset E'$. To prove this, suppose that $E \setminus E' \neq \emptyset$ (isolated points). Let $x \in E \setminus E'$. Since x is an isolated point of E, $\exists \delta > 0$ s.t $B(x, \delta) \cap E = \{x\}$. So $B(x, \delta) \setminus \{x\} = (x-\delta, x) \cup (x, x+\delta) \subset E^c$. This implies that there exists (a_i, b_i) and (a_j, b_j) which have the common edge point $\{x\}$. This contradicts to the assumption. Now the proof of $\Leftarrow$ is complete.

□

53 (Example 19) Suppose that $E \stackrel{\text{def}}{=} \{x_n\}_{n \geqq 1}$ (i.e E is a countable set) and we derive a contradiction.

STEP 1. (pick y_1, δ_1) Let us pick $y_1 \in E \setminus \{x_1\}$. Let us take $\delta_1 \in (0, |x_1 - y_1|)$.

STEP 2. (pick y_2, δ_2) Note that $B(y_1, \delta_1) \cap E \setminus \{y_1\}$ is not empty because $y_1 \in E$ and a perfect set E does not have an isolated point. We can pick $y_2 \in B(y_1, \delta_1) \cap E \setminus \{y_1\} (\neq \emptyset)$ with $y_2 \neq x_2$. (Otherwise, it follows that $B(y_1, \delta_1) \cap E \setminus \{y_1\} = \{x_2\}$. If we take $\delta_1^* < |x_2 - y_1|$, then $B(y_1, \delta_1^*) \cap E \setminus \{y_1\} = \emptyset$, hence y_1 is an isolated point of E, which contradicts to the fact that a perfect set E has no isolated point.) Let us take δ_2 with $0 < \delta_2 < |y_2 - x_2|$.

STEP 3. (pick y_3, δ_3) Let us continue the same procedure. Note that $B(y_2, \delta_2) \cap E \setminus \{y_2\}$ is not empty because $y_2 \in E$ and E is a perfect set (it has no isolated point). We can pick $y_3 \in B(y_2, \delta_2) \cap E \setminus \{y_2\}$ with $y_3 \neq x_3$. (Otherwise, it follows that $B(y_2, \delta_2) \cap E \setminus \{y_2\} = \{x_3\}$ and if we change δ_2 change to $\delta_2^* \in (0, |x_3 - y_2|)$, then y_2 turns out to be an isolated point.) Let us take δ_3 with $0 < \delta_3 < |y_3 - x_3|$.

STEP 4. (derive contradiction) By continuing the same procedure, we obtain $B(y_n, \delta_n)$. Note that $\overline{B}(y_n, \delta_n) \cap E$ is a bounded and no-empty closed set. Let $F_n \stackrel{\text{def}}{=} \bigcap_{m=1}^{n} \overline{B}(y_m, \delta_m) \cap E$. Then $F_{n+1} \subset F_n$ and F_n is also a bounded closed set. So

$$\bigcap_{n=1}^{\infty} \overline{B}(y_n, \delta_n) \cap E = \bigcap_{n=1}^{\infty} F_n \neq \emptyset,$$

by Theorem 1.17 Cantor's Intersection Theorem.

However, let us recall that $B(y_n, \delta_n)$ does not contain $\{x_n\}$ because $\delta_n < |x_n - y_n|$. So the $\bigcap_{n=1}^{\infty} \overline{B}(y_n, \delta_n)$ does not contain any x_n ($n \geqq 1$). Hence the $\bigcap_{n=1}^{\infty} \overline{B}(y_n, \delta_n) \cap E = \emptyset$ and it contradicts to the fact above.

□

54 (Exercise 1) Fix $x \in E$. Let us consider $A \stackrel{\text{def}}{=} \{x - y \mid y \in E\}$. Obviously, A is an uncountable set because E is uncountable. (There is a bijective mapping between A

and E, so the cardinality is the same.) So $A \subset \mathbb{Q}$ can not happen. We can pick $a \in A \setminus \mathbb{Q}$. Then $x - a \in E$ is the desired y. $\square$

55 (Exercise 4) Let C be a Cantor set defined on $[0,1]$. In constructing C_{n+1} we remove 2^n intevals from C_n. Let $\{J_{n,k}\}_{k=1}^{2^n}$ be the intervals that are removed from C_n to construct C_{n+1}. Let $c_{n,k}$ be the center of $J_{n,k}$ and define $E \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^n} c_{n,k}$. Then E is the desired set. Obviously, each $x \in E$ is an isolated point of E. We prove that $E' = C$. (C is a perfect set.)

STEP 1. ($C \subset E'$) Let $\delta > 0$ be an arbitrarily small positive number and let us pick arbitrary point $x \in E$. Let us recall that $x \in C_n$ for all n. We take large $n \in \mathbb{N}$ so that $\frac{1}{2} \cdot \frac{1}{3^n} < \delta$. We can find k s.t $x \in I_{n,k}$. In constructing C_{n+1}, the middle part of $I_{n,k}$ is also removed. Let the removed interval be J_{n,k^*}. Then $|c_{n,k^*}| \leqq \frac{1}{2} \cdot \frac{1}{3^n} < \delta$. This implies that we can find a sequence of $\{c_n\} \subset E$ s.t $c_n \to x$. So $x \in E'$.

STEP 2. ($E' \subset C$) Let $x \in E'$. There exists $\{x_n\}_{n \geqq 1} \subset E$ s.t $x_n \to x$. First, $|x_n - x_{n'}| \leqq |x_n - x| + |x_{n'} - x| \to 0$ as $n, n' \to \infty$, so $\{x_n\}$ is a Cauchy sequence. We can take a subsequence $\{n_m\}$ s.t

$$|x_{n(m)} - x_{n(m+1)}| \leqq \frac{1}{2} \cdot \frac{1}{3^m}$$

For simplicity, we define $x_m^* \overset{\text{def}}{=} x_{n(m)}$. Then $|x_m^* - x_{m+1}^*| \leqq \frac{1}{2} \cdot \frac{1}{3^m}$ and note that $x_m^* \to x$.

Let us recall that x_m^* is a center of an interval J which is removed when constructing C. If the length of J is larger than $\frac{1}{3^m}$, then the above inequality does not hold. So if $J_{n(m),k(m)}$ is the interval that contains x_m^*, then the index number $n(m) > m$. Let us consider the intervals $I_{n(m)+1,*}$ beside to $J_{n(m),k(m)}$ which are the components of $C_{n(m)+1}$. Note that the edge points of $I_{n(m)+1,*}$ are contained in C. So we pick the nearest one from x_m^*, and denote it as y_m. Then $|x_m^* - y_m| \leqq \frac{1}{2} \cdot \frac{1}{3^m}$. Finally, $|y_m - x| \leqq |y_m - x_m^*| + |x_m^* - x| \leqq \frac{1}{3^m} \to 0$. So we have $\{y_m\}_{m \geqq 1} \subset C$ s.t $y_m \to x$. This implies that $x \in C' = C$.

$\square$

§ 1.3

56 (Definition 1.33 and Theorem 1.24)

(1)

$$\operatorname{dist}(E_1, E_2) \overset{\text{def}}{=} \inf_{y_1 \in E_1, y_2 \in E_2} \{|y_1 - y_2|\}.$$

(2) By the definition of $\operatorname{dist}(x_0, F)$, we can find a sequence of points $\{y_n\} \subset F$ s.t $|x_0 - y_n| \to \operatorname{dist}(x_0, F)$. Obviously $\operatorname{dist}(x_0, F) < \infty$. (If we arbitrarily pick $y \in F$, then $|x_0 - y| < \infty$) Let $d \overset{\text{def}}{=} \operatorname{dist}(x_0, F)$. Without loss of generality, we may suppose that $|x_0 - y_n| < d + \frac{1}{n}$. By triangular inequality, we have $|y_n| \leqq |x_0| + |y_n - x_0| \leqq |x_0| + d + 1 < \infty$. So $\{y_n\}_{n \geqq 1} \subset F$ is bounded. By Bolzano-Weierstrass's Theorem, we have a subsequence n_k s.t $y_{n_k} \to y_0 \in F' \subset F$. Finally $|x_0 - y_0| \leqq |x_0 - y_{n_k}| + |y_{n_k} - y_0| < d + \frac{1}{n_k} + |y_{n_k} - y_0| \to d$.

□

57 (Theorem 1.25) Let $x, y \in \mathbb{R}^d$. Let $\epsilon > 0$ be an arbitrary positive number. By the definition of $\text{dist}(y, E)$, we can find $e \in E$ s.t $|y - e| < \text{dist}(y, E) + \epsilon$. Since $\text{dist}(x, E) \leqq |x - e| \leqq |x - y| + |y - e| < |x - y| + \text{dist}(y, E) + \epsilon$. So we habe $\text{dist}(x, E) - \text{dist}(y, E) < |x - y| + \epsilon$. Therefore $\text{dist}(x, E) - \text{dist}(y, E) \leqq |x - y|$. By swapping x, y we have $|\text{dist}(x, E) - \text{dist}(y, E)| \leqq |x - y|$. This implies that $\exists \delta = \epsilon$ s.t $\forall \epsilon > 0, \forall x, y \in \mathbb{R}^d$ with $|x - y| < \delta$, $|\text{dist}(x, E) - \text{dist}(y, E)| < \epsilon$. Now the proof is complete. □

58 (Corollary 1.26) Suppose that F_1 is bounded. There exist sequences $\{x_{1,k}\} \subset F_1, \{x_{2,k}\} \subset F_2$ s.t

$$|x_{1,k} - x_{2,k}| \to \text{dist}(F_1, F_2) < \infty.$$

Without loss of generality, we may suppose that

$$|x_{1,k} - x_{2,k}| < \text{dist}(F_1, F_2) + \frac{1}{k}.$$

By assumption, $\{x_{1,k}\}$ is bounded ($|x_{1,k}| \leqq M_1$), so by Bolzano-Weierstrass's theorem, there exists a subsequnce k_ℓ s.t $x_{1,k_\ell} \to x_0 \in F_1' \subset F_1$. By triangular inequality, $|x_{2,k_\ell}| \leqq |x_{2,k_\ell} - x_{1,k_\ell}| + |x_{1,k_\ell}| \leqq \text{dist}(F_1, F_2) + 1 + M_1$, so x_{2,k_ℓ} is also bounded. Again by Bolzano-Weierstrass's Theorem, there exists a further subsequence k_{ℓ_m} s.t $x_{2,k_{\ell_m}} \to x_2 \in F_2' \subset F_2$. Finally, $\text{dist}(F_1, F_2) \leqq |x_1 - x_2| \leqq \left|x_{1,k_{\ell_m}} - x_1\right| + \left|x_{2,k_{\ell_m}} - x_2\right| + \left|x_{1,k_{\ell_m}} - x_{2,k_{\ell_m}}\right| \to \text{dist}(F_1, F_2)$. Now the proof is complete. □

59 (Example 2)

$$f(x) \stackrel{\text{def}}{=} \frac{\text{dist}(x, F_2)}{\text{dist}(x, F_1) + \text{dist}(x, F_2)}.$$

Notice. $\text{dist}(x, F_1) + \text{dist}(x, F_2) \neq 0$ because if $\text{dist}(x, F_1) = 0, \text{dist}(x, F_2) = 0$, then $x \in F_1, x \in F_2$. (By Theorem 1.24) However, F_1, F_2 are disjoint. □

60 (Theorem 1.27)

STEP 1. ($g_1(x)$) Let us divide F into the following three parts.

- $A_1 = \{x \in F \mid M/3 \leqq f(x) \leqq M\}$.
- $B_1 = \{x \in F \mid -M \leqq f(x) \leqq -M/3\}$.
- $C_1 = \{x \in F \mid -M/3 < f(x) < M/3\}$.

case 1. ($A_1, B_1 \neq \emptyset$) Let us define

$$g_1(x) \stackrel{\text{def}}{=} \frac{M}{3} \cdot \frac{\text{dist}(x, A_1) - \text{dist}(x, B_1)}{\text{dist}(x, A_1) + \text{dist}(x, B_1)}.$$

We claim that

- $g_1(x)$ is continuous on $\mathbb{R}^d$. (Of course, well-defined. i.e $\text{dist}(x, A_1) + \text{dist}(x, B_1) \neq 0$)
- $|g_1(x)| \leqq \frac{M}{3}$ on $\mathbb{R}^d$.
- $|f(x) - g_1(x)| \leqq \frac{2M}{3}$ on F.

Continuity of $g_1(x)$ is shown using Theorem 1.25. When $x \in A_1$, $M/3 \leqq f(x) \leqq M$ and $g_1(x) = M/3$, so $0 \leqq f(x) - g_1(x) \leqq 2M/3$. When $x \in B_1$, the proof is similar. When $x \in \mathbb{R}^d \setminus (A_1 \cup B_1)$, $-M/3 \leqq g_1(x) \leqq M/3$. Of course, $x \in C_1 \subset \mathbb{R}^d \setminus (A_1 \cup B_1)$, $-M/3 \leqq g_1(x) \leqq M/3$ holds, hence $|f(x) - g_1(x)| \leqq 2M/3$.

case 2. ($A_1 \neq \emptyset$, $B_1 = \emptyset$) Let us define

$$g_1(x) \stackrel{\text{def}}{=} \frac{M}{3}.$$

Note that $g_1(x)$ is continuous on $\mathbb{R}^d$, $|g_1(x)| \leqq \frac{M}{3}$ on $\mathbb{R}^d$ and $|f(x) - g_1(x)| \leqq \frac{2M}{3}$ on F. The proof is easy. (We show that last part.) Since B_1 is empty, $-M/3 < f(x) \leqq M$ for all $x \in F$. Therefore $|f(x) - g_1(x)| \leqq 2M/3$ on F.

case 3. ($A_1 = \emptyset$, $B_1 \neq \emptyset$) Let us define

$$g_1(x) \stackrel{\text{def}}{=} -\frac{M}{3}.$$

Note that $g_1(x)$ is continuous on $\mathbb{R}^d$, $|g_1(x)| \leqq \frac{M}{3}$ on $\mathbb{R}^d$ and $|f(x) - g_1(x)| \leqq \frac{2M}{3}$ on F. The proof is completely same as the previous one.

case 4. ($A_1, B_1 = \emptyset$) Let us define

$$g_1(x) \stackrel{\text{def}}{=} 0.$$

Note that $g_1(x)$ is continuous on $\mathbb{R}^d$, $|g_1(x)| \leqq \frac{M}{3}$ on $\mathbb{R}^d$ and $|f(x) - g_1(x)| \leqq \frac{2M}{3}$ on F. The proof is easy. (We show the last part.) Since both A_1, B_1 are empty, this implies that $-M/3 < f(x) < M/3$ on F. So $|f(x) - g_1(x)| = |f(x)| < M/3 \leqq 2M/3$ on F.

In conslusion, we can find a function $g_1(x)$ defined on $\mathbb{R}^d$ s.t

- $g_1(x) \in C(\mathbb{R}^d)$,
- $|g_1(x)| \leqq M/3$ on $\mathbb{R}^d$,
- $|f(x) - g_1(x)| \leqq 2M/3$ on F.

STEP 2. ($g_2(x)$) Let $\tilde{f}(x) \stackrel{\text{def}}{=} f(x) - g_1(x)$ and let us repeat the similar argument with the previous step. Let us divide F into the following three parts.

- $A_2 = \{x \in F \mid 2M/9 \leqq \tilde{f}(x) \leqq 2M/3\}$.
- $B_2 = \{x \in F \mid -2M/3 \leqq \tilde{f}(x) \leqq -2M/9\}$.
- $C_2 = \{x \in F \mid -2M/9 < \tilde{f}(x) < 2M/9\}$.

case 1. ($A_2, B_2 \neq \emptyset$) Let us define

$$g_2(x) \stackrel{\text{def}}{=} \frac{2M}{9} \cdot \frac{\operatorname{dist}(x, A_2) - \operatorname{dist}(x, B_2)}{\operatorname{dist}(x, A_2) + \operatorname{dist}(x, B_2)}.$$

case 2. ($A_2 \neq \emptyset$, $B_2 = \emptyset$) Let us define

$$g_2(x) \stackrel{\text{def}}{=} \frac{2M}{9}.$$

case 3. ($A_2 = \emptyset$, $B_2 \neq \emptyset$) Let us define

$$g_2(x) \stackrel{\text{def}}{=} -\frac{2M}{9}.$$

case 4. ($A_2 \neq \emptyset$, $B_2 = \emptyset$) Let us define

$$g_2(x) \stackrel{\text{def}}{=} 0.$$

In this way, we have a function defined on $g_2(x)$ s.t

- $g_2(x) \in C(\mathbb{R}^d)$,
- $|g_2(x)| \leqq 2M/9$ on $\mathbb{R}^d$,
- $|\tilde{f}(x) - g_2(x)| = |f(x) - g_1(x) - g_2(x)| \leqq (2/3)^2 \cdot M$ on F.

STEP 3. ($g(x)$) From the arguments above, we can obtain a sequence of functions $\{g_n(x)\}$ satisfying

- $g_n(x) \in C(\mathbb{R}^d)$,
- $|g_n(x)| \leqq 1/3 \cdot (2/3)^{n-1} \cdot M$ on $\mathbb{R}^d$,
- $|f(x) - \sum_{k=1}^n g_k(x)| \leqq (2/3)^n \cdot M$ on F.

We prove that

$$g(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} g_n(x)$$

is the desired continuous function on $\mathbb{R}^d$.

First, we prove that $\sum_{n=1}^{\infty} g_n(x)$ converges (the limit exists and is finite). Note that

$$\begin{aligned}\sum_{n=1}^{\infty} |g_n(x)| &\leqq \sum_{n=1}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} \cdot M \\ &= M, \ (\forall x \in \mathbb{R}^d) \ \cdots (*).\end{aligned}$$

Since absolute convergence implies convergence, (i.e $\sum_{n=1}^{\infty} |a_n| < \infty \Rightarrow \sum_{n=1}^{\infty} a_n$ exists and finite.) $g(x)$ is well-defined and is finite. Therefore,

$$\begin{aligned}|f(x) - g(x)| &= \lim_{n\to\infty} \left| f(x) - \sum_{k=1}^{n} g_k(x) \right| \\ &\leqq \lim_{n\to\infty} \left(\frac{2}{3}\right)^n \cdot M = 0.\end{aligned}$$

From (∗), we also have

$$|g(x)| \leqq \sum_{n=1}^{\infty} |g_n(x)| \leqq M < \infty, \ (\forall x \in \mathbb{R}^d).$$

Second, we prove that $g(x)$ is continuous on $\mathbb{R}^d$. Let $G_n(x) \stackrel{\text{def}}{=} \sum_{k=1}^{n} g_k(x)$. Since $G_n(x)$ is a finite sum of continuous functions, $G_n(x)$ is continuous on $\mathbb{R}^d$. We prove that $G_n(x) \xrightarrow{u} g(x)$ (converges uniformly) on $\mathbb{R}^d$. Then $g(x)$ is continuous on $\mathbb{R}^d$. (Let us recall that if $f_n(x)$ is continuous and $f_n(x) \xrightarrow{u} f(x)$, then $f(x)$ is also continuous.)

$$\begin{aligned}
|G_n(x) - g(x)| &= |G_n(x) - \lim_{m\to\infty} G_m(x)| \\
&= \lim_{m\to\infty} |G_n(x) - G_m(x)| \\
&= \lim_{m\to\infty} \left| \sum_{i=1}^{n} g_i(x) - \sum_{i=1}^{m} g_i(x) \right| \\
&\leqq \lim_{m\to\infty} \left| \sum_{i=n+1}^{m} g_i(x) \right| \\
&\leqq \lim_{m\to\infty} \sum_{i=n+1}^{m} |g_i(x)| \\
&\leqq \lim_{m\to\infty} \sum_{i=n+1}^{m} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{i-1} \cdot M \\
&= \left(\frac{2}{3}\right)^{n} \cdot M, \ (\forall x \in \mathbb{R}^d)
\end{aligned}$$

From the inequality above, we have

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{R}^d} |G_n(x) - g(x)| \leqq \lim_{n\to\infty} \left(\frac{2}{3}\right)^{n} \cdot M = 0$$

So $G_n(x) \xrightarrow{u} g(x)$ on $\mathbb{R}^d$ and we conclude that $g(x)$ is continuous on $\mathbb{R}^d$.

□

61 (Extension of Theorem 1.27) Let

$$f^*(x) \stackrel{\text{def}}{=} \arctan f(x).$$

Note that $f^*(x)$ is continuous and bounded on F. We apply Theorem 1.27 to $f^*(x)$ and then obtain $g^*(x) \in C(\mathbb{R}^d)$ with $f^*(x) = g^*(x)$ on F. Let

$$g(x) \stackrel{\text{def}}{=} \tan g^*(x).$$

Then $g(x) \in C(\mathbb{R}^d)$ and $f(x) = g(x)$. So $g(x)$ is the desired function. (We should prove that $g^*(x) \neq \pm\frac{\pi}{2}$. We will supplement the proof in the future.) □

62 (Exercise 1) We show that $E' \subset E$ (so $E' \backslash E = \emptyset$). We suppose that $E' \backslash E \neq \emptyset$ and derive a contradiction. Let $x \in E' \setminus E$. Since $x \in E'$, there exists $\{y_n\}_{n \geqq 1} \subset E$ s.t $y_n \to x$ as $n \to \infty$. Since $x \in E^c$, there exists $y \in E$ s.t $|x - y| = \mathrm{dist}(x, E)$. Note that

$$0 \leqq |x - y| = \mathrm{dist}(x, E) \leqq |x - y_n|,$$

because $y_n \in E$. By taking $n \to \infty$, we have $|x - y| = 0$. This implies that $x = y$. $x \in E^c$ but $y \in E$. (contradiction!!) Now the proof is complete. □

63 (Exercise 2) Let us recall Corollary 1.26. Apply Corollary 1.26 to the closed sets F and G^c. There exists $x_1 \in F$ and $x_2 \in G^c$ s.t

$$|x_1 - x_2| = \mathrm{dist}(F, G^c).$$

Let us take $r \overset{\text{def}}{=} |x_1 - x_2|$ and let $x \in \mathbb{R}^d$. If $\mathrm{dist}(x, F) < r$, then $x \notin G^c$ because if $x \in G^c$, then $\mathrm{dist}(x, F) \geqq r$. So $\mathrm{dist}(x, F) < r \Rightarrow x \notin G^c \Leftrightarrow x \in G$. Now we conclude that $\{x \in \mathbb{R}^d \mid \mathrm{dist}(x, F) < r\} \subset G$. □

§ 1.4

64 (Exercise 8) Let us pick $y^* = f(x^*) \in E$. By assumption, we have $\delta^* > 0$ s.t $f(x) \geqq f(x^*)$ for all $x \in (x^* - \delta^*, x^* + \delta^*)$. So $f(x)$ takes the minimum value at $x = x^*$ if $x \in (x^* - \delta^*, x^* + \delta^*)$. We can find and choose $r_1^*, r_2^* \in \mathbb{Q}$ s.t $x^* \in (r_1^*, r_2^*) \subset (x^* - \delta^*, x^* + \delta^*)$. Now we have a map $\{y^*\}_{y^* \in E} \mapsto \{(r_1^*, r_2^*)\}$. Conversely, if we are given (r_1^*, r_2^*), then $f(x)$ takes the minimum value at some $x^* \in (r_1^*, r_2^*)$, so we can determine $y^* = f(x^*)$. So $y^* \in E$ and (r_1^*, r_2^*) are one-to-one. Obviously, there exists only a countable number of (r_1^*, r_2^*), hence E is also countable. □

65 (Exercise 9) Let us define the surface

$$S(x, r) \overset{\text{def}}{=} \{y \in \mathbb{R}^d \mid |x - y| = r\}, (d = 3).$$

First, let us pick $x_1 \in E$ and note that

$$E = \bigcup_{r_1 \in \mathbb{Q} \cap [0, \infty)} S(x_1, r_1) \cap E,$$

because the distance of any two points is rational number. ($\mathbb{Q}$: a collection of all rational number.) From the discussion above, it is enough for us to prove that $S(x_1, r_1) \cap E$ is at most countable (countable of finite) for each $r_1 \in \mathbb{Q} \cap [0, \infty)$.

Second, let us pick $x_2 \in S(x_1, r_1) \cap E$ with $x_2 \neq x_1$. (If we fail to choose such x_2, this means that $S(x_1, r_1) \cap E = \{x_1\}$. And it is a finite set.) Note that

$$S(x_1, r_1) \cap E = \bigcup_{r_2 \in \mathbb{Q} \cap [0, \infty)} S(x_2, r_2) \cap S(x_1, r_1) \cap E.$$

So it is enough for us to prove that $S(x_2, r_2) \cap S(x_1, r_1) \cap E$ is at most countable for each $r_2 \in \mathbb{Q} \cap [0, \infty)$.

Third, let us pick $x_3 \in S(x_2, r_2) \cap S(x_1, r_1) \cap E$ with $x_3 \neq x_2, x_1$. (If we fail to choose such x_3, this means that $S(x_2, r_2) \cap S(x_1, r_1) \cap E \subset \{x_1, x_2\}$, hence it is finite.) Note that

$$S(x_2, r_2) \cap S(x_1, r_1) \cap E = \bigcup_{r_3 \in \mathbb{Q} \cap [0,\infty)} S(x_3, r_3) \cap S(x_2, r_2) \cap S(x_1, r_1) \cap E.$$

The right hand side are intersection points of three surfaces. The number of intersection points of three surfaces are at most 2. Now the proof is complete. □

66 (Exercise 11) □

67 (Exercise 13) We show that $E' \subset E$. When $E' = \emptyset$, the statement holds obviously, we suppose that $E' \neq \emptyset$. Let us fix $\epsilon > 0$ which is an arbitrary positive number. Let us take $x \in E'$. There exists $\{x_n\} \subset E$ s.t $x_n \to x$. For sufficiently large n, $|x_n - x| < \frac{\epsilon}{2}$. Then $x + \epsilon = x - x_n + x_n + \epsilon > -\frac{\epsilon}{2} + x_n + \epsilon = x_n + \frac{\epsilon}{2}$, and $x - \epsilon = x - x_n + x_n - \epsilon < \frac{\epsilon}{2} + x_n - \epsilon = x_n - \frac{\epsilon}{2}$. Since $f(x)$ is monotone increasing, $f(x+\epsilon) - f(x-\epsilon) \geqq f(x_n + \epsilon/2) - f(x_n - \epsilon/2) > 0$, because $x_n \in E$ hence $\forall \epsilon^* (= \epsilon/2)$, $f(x_n + \epsilon^*) - f(x_n - \epsilon^*) > 0$ So $x \in E$. Now we conclude that $E' \subset E$ and the proof is complete. □

68 (Exercise 14.1) E is an infinite set, and $E \subset F$ implies that E is also bounded. By Bolzano-Weierstrass Theorem, E has at least one limit point. So $E' \neq \emptyset$. And $E' \subset F' = F$. So $E' \cap F \neq \emptyset$. □

69 (Exercise 14.2)

STEP 1. (F is closed) Let us pick $x \in F'$. Then there exists $\{x_n\}_{n \geqq 1} \subset F$ s.t $x_n \to x$ ($x_i \neq x_j$ if $i \neq j$). Let $E \stackrel{\text{def}}{=} \{x_n\}$. And $E' = \{x\}$, $\cdots (*)$. So $E' \cap F \neq \emptyset$ implies that $x \in F$, and we conclude that F is closed.

$(*)$ It is easy to show that y $(\neq x)$ can not be $y \in E'$. For sufficiently large $n > N$, $|x_n - x| < \frac{|x-y|}{2}$. By triangular inequality, $|x_n - y| \geqq |x - y| - |x_n - x| \geqq \frac{|x-y|}{2} > 0$. Now let $\delta \stackrel{\text{def}}{=} \min\{|x_1 - y|, \cdots, |x_N - y|, |x-y|/2\}$, and then $B(y, \delta) \setminus \{y\} \cap \{x_n\} = \emptyset$.

STEP 2. (F is bounded) Suppose that F is not bounded. Then we can take $\{x_n\} \subset F$ s.t $|x_n| \to \infty$. Let $E \stackrel{\text{def}}{=} \{x_n\}$. Then $E' = \emptyset$, and it contradicts to the assumption. So F is bounded.

□

70 (Exercise 15) We show that $E' \subset E$. If $E' = \emptyset$, then the statement holds immediately, so we assume that $E' \neq$. Let us take $t \in E'$. There exists $\{t_n\} \subset E$ s.t $t_n \to t$ as $n \to \infty$. By assumption, there exists $x_n \in F$ s.t $|t_n - x_n| = F$. When n is sufficiently large (say $n \geqq N$ for some $N \in \mathbb{N}$), $|t_n - t| \leqq \delta$ for some $\delta > 0$. So $|t_n| \leqq |t_n - t| + |t| \leqq \delta + |t| < \infty$ for all $n \geqq N$. Therefore, we may suppose that $|t_n|$ is bounded. And $|x_n| = |x_n - t_n + t_n| \leqq |t_n - x_n| + |t_n| = r + |t_n| \leqq r + \delta + |t| < \infty$. So $\{x_n\} \subset F$ is bounded. By Bolzano-Weierstrass' Theorem, we can find a subsequence n_k

s.t $x_{n_k} \to x$, and $x \in F$ because F is closed. Note that

$$\begin{aligned} |t-x| &= |t - t_{n_k} + t_{n_k} - x_{n_k} + x_{n_k} - x| \\ &\leqq |t - t_{n_k}| + |t_{n_k} - x_{n_k}| + |x - x_{n_l}| \\ &= |t - t_{n_k}| + r + |x - x_{n_k}| \to r, \text{ as } k \to \infty \end{aligned}$$

and

$$\begin{aligned} |t-x| &= |t - t_{n_k} + t_{n_k} - x_{n_k} + x_{n_k} - x| \\ &\geqq -|t - t_{n_k}| + |t_{n_k} - x_{n_k}| - |x - x_{n_l}| \\ &= -|t - t_{n_k}| + r - |x - x_{n_k}| \to r, \text{ as } k \to \infty \end{aligned}$$

(The inequalities above are obtained by triangular inequality. $|a+b| \leqq |a| + |b|$ and $|a+b| \geqq |a| - |b|$. Moreover $|a+b+c| \leqq |a| + |b| + |c|$ and $|a+b+c| \geqq -|a| + |b| - |c|$.) Now we have $t \in E$. So $E' \subset E$ and we conclude that E is a closed set. □

71 (Exercise 17) Let us fix $y \in \mathbb{R}$. If E_y is an empty set, then $E'_y = \emptyset \subset E_y = \emptyset$, so the statement holds. Suppose that $E_y \neq \emptyset$. We prove that $E'_y \subset E_y$. If $E'_y = \emptyset$, then $E'_y \subset E_y$. So E'_y is closed. We assume that $E'_y \neq \emptyset$. Let us pick $x \in E'_y$. Then we have $\{x_n\} \subset E_y$ s.t $x_n \to x$. By definition, $(x_n, y) \in E$. Note that $(x_n, y) \to (x, y)$ and $(x, y) \in E$ because E is a closed set. This implies that $x \in E_y$. So we have $E'_y \subset E_y$ and we conclude that E_y is closed. □

72 (Exercise 18)

STEP 1. ($\subset$) Note that $f\left(\bigcap_{k=1}^{\infty} F_k\right) \subset f(F_k)$ for all $k \geqq 1$. Therefore,

$$f\left(\bigcap_{k=1}^{\infty} F_k\right) \subset \bigcap_{k=1}^{\infty} f(F_k)$$

holds immediately.

STEP 2. ($\supset$) Let us pick $y_0 \in \bigcap_{k=1}^{\infty} f(F_k)$. Then $y_0 \in f(F_k)$ for all $k \geqq 1$. There exists $x_k \in F_k$ s.t $f(x_k) = y_0$. Since $\{x_k\} \subset F_1$ (because F_k is a decreasing sequence) and F_1 is bounded and closed, we can find a subsequence k_ℓ s.t $x_{k_\ell} \to x_0 \in F_1$. Note that $x_{k_\ell} \in F_2$ if $\ell \geqq 2$, so $x_{k_\ell} \to x_0 \in F_2$. By repeating the same arguments, we conclude that $x_0 \in F_k$ for all $k \geqq 1$. So $x_0 \in \bigcap_{k=1}^{\infty} F_k$, hence $f(x_0) \in f\left(\bigcap_{k=1}^{\infty} F_k\right)$. Moreover, $f(x_{k_\ell}) \to f(x_0)$ because $f(x) \in C(\mathbb{R})$, and $f(x_{k_\ell}) = y_0$ for all $\ell \geqq 1$, so $f(x_0) = y_0$. We conclude that $y_0 \in f(\bigcap_{k=1}^{\infty} F_k)$.

□

73 (Exercise 19)

STEP 1. We prove that

$$E_1 \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid f(x) > t\}, \; E_2 \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid f(x) < t\}$$

are open for all $t \in \mathbb{R}$. (See Example 2 and 6.) However it is enough for us to prove that

$$E_1 \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid f(x) > r\}, \; E_2 \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid f(x) < r\}$$

are open for all $r \in \mathbb{Q}$. This is because for all $t \in \mathbb{R}$, we can find a sequence of $r_n \in \mathbb{Q}$ s.t $r_n \searrow t$ (or $r_n \nearrow t$ for E_2), hence

$$E_1 \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid f(x) > t\} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} \mid f(x) > r_n\}.$$

(Note that a countable union of open sets is also open.)

STEP 2. Let $r \in \mathbb{Q}$ be an arbitrary rational number and let us fix r. Let $E_1 \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid f(x) > r\}, E_2 \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid f(x) < r\}$. By assumption, $E_1 \cup E_2$ is open. We prove that E_1 and E_2 are also open. Let us pick $x_0 \in E_1$. Since $x_0 \in E_1 \cup E_2$ and $E_1 \cup E_2$ is open, there exists $\delta_0 > 0$ s.t $B(x_0, \delta_0) \subset E_1 \cup E_2$.

STEP 3. Suppose that $B(x_0, \delta_0) \cap E_2 \neq \emptyset$. Let us pick $y_0 \in B(x_0, \delta_0) \cap E_2$. Note that $|x_0 - y_0| < \delta$. Since $f(x_0) > r$ and $f(y_0) < r$, there exists $z \in B(x_0, |x_0 - y_0|)$ s.t $f(z) = r$ by assumption. However, $z \in B(x_0, \delta_0) \subset E_1 \cup E_2$. So $f(z) > r$ or $f(z) < r$. This contradicts to the fact that $f(z) = r$. Therefore, $B(x_0, \delta_0) \cap E_2 = \emptyset$, hence $B(x_0, \delta_0) \subset E_1$. So E_1 is open. Similarly, E_2 is also open.

□

74 (Exercise 20) Let $x \in \overline{E}_1$ and let $y \in E_2'$. Note that $\overline{E}_1 = E_1 \cup E_1'$.

case 1. ($x \in E_1$ and $y \in E_1'$) There exists $\{y_n\} \subset E_1$ ($y_i \neq y_j$ if $i \neq j$) s.t $y_n \to y$. $\{x + y_n\} \subset E_1 + E_2$ and $x + y_n \to x + y$. And $x + y_i \neq x + y_j$ if $i \neq j$. So $x + y \in (E_1 + E_2)'$.

case 2. ($x \in E_1'$ and $y \in E_1'$) There exist $\{x_n\} \subset E_1$ and $\{y_n\} \subset E_2$ s.t $x_n \to x$ and $y_n \to y$. $\{x_n + y_n\} \subset E_1 + E_2$ and $x_n + y_n \to x + y$. However, we have to consider the case $x_n + y_n = x + y$ for all $n > N$ where N is some integer. In such a case, we can consider $\{x_n + y_{n+1}\}$. Then $x_n + y_{n+1}$ are different from each other for sufficiently large n. (Let $a = x_n + y_n$ for $n > N$. $x_n + y_{n+1} = x_n + y_{n+1} - a + a = y_{n+1} - y_n + a$. And note that $y_{n+1} - y_n \neq 0$ but $y_{n+1} - y_n \to 0$.) From this argument, $x_n + y_{n+1} \to x + y \in (E_1 + E_2)'$. Now the proof is complete.

□

75 (Exercise 21) $\partial E = \emptyset$ implies that $\overline{E} = \mathring{E} = E$. From this relationship, E is both open and closed. We prove that if E is open and closed (hence E^c is also open and closed), then $E = \mathbb{R}^d$ or $\emptyset$. We suppose that $E \neq \emptyset$ and $E \neq \mathbb{R}^d$. We can take $x \in E$ and $y \in E^c$. (We will supplement the proof in the future.) □

76 (Exercise 22) We suppose that $G_1 \cap \overline{G}_2 \neq \emptyset$ and derive a contradiction. We can take $x_0 \in G_1 \cap \overline{G}_2$. Then $x_0 \in G_1$. So $x_0 \notin G_2$ because G_1 and G_2 are disjoint, and note that $x_0 \in \overline{G}_2$, so $x_0 \in G_2'$. We can take $\{x_n\} \subset G_2$ s.t $x_n \to x_0$. Since G_1 is open, there exists $\delta > 0$ s.t $B(x_0, \delta) \subset G_1$. When n is sufficiently large, $|x_n - x_0| < \delta$. So $x_n \in B(x_0, \delta) \subset G_1$. This contradicts to the assumption that $\{x_n\} \subset G_2$. □

77 (Exercise 23) Let $E \stackrel{\text{def}}{=} G^c = \mathbb{R}^d \setminus E$. Then $G \cap \overline{G^c} \subset \overline{(G \cap G^c)} = \emptyset$. So $G \cap \overline{G^c} = \emptyset$. This implies that G and $\overline{G^c}$ are disjoint, hence $\overline{G^c} \subset G^c$. Therefore G^c is closed. So we conclude that G is open. $\square$

78 (Exercise 25)

STEP 1. ($\Rightarrow$) Suppose that $f(x) \in C(\mathbb{R})$. We prove that G_1, G_2 are open. The procedure of proof is similar for G_1 and G_2, we only prove that G_1 is open. Let us pick $(x_0, y_0) \in G_1$, then $y_0 < f(x_0)$. Since $f(x)$ is continuous at $x = x_0$, there exists $\delta_0 > 0$ s.t $|f(x) - f(x_0)| < \epsilon_0 \overset{\text{def}}{=} \frac{f(x_0)-y_0}{2}$, $\forall x \in B(x_0, \delta_0)$. Especially, we have

$$f(x_0) - \epsilon_0 < f(x),\ \forall x \in B(x_0, \delta_0), \cdots\ (*)$$

Let $r \overset{\text{def}}{=} \min\{\delta_0, \epsilon_0\}$. We claim that $B((x_0, y_0), r) \subset G_1$, hence G_1 is open. Let us pick an arbitrary point $(x, y) \in B((x_0, y_0), r)$.

$$\begin{aligned} y < y_0 + r &\overset{(*1)}{=} (f(x_0) - 2\epsilon_0) + r \\ &= (f(x_0) - \epsilon_0) + (r - \epsilon_0) \\ &\overset{(*2)}{\leqq} f(x_0) - \epsilon_0 + 0 \\ &\overset{(*3)}{<} f(x) \end{aligned}$$

- $(*1)$ $\epsilon_0 = \frac{f(x_0)-y_0}{2}$ by definition. So $y_0 = f(x_0) - 2\epsilon_0$.
- $(*2)$ $r = \min\{\delta_0, \epsilon_0\} \leqq \epsilon_0$.
- $(*3)$ See $(*)$.

So $y < f(x)$ and we conclude that $(x, y) \in G_1$ for all $(x, y) \in B((x_0, y_0), r)$.

STEP 2. ($\Leftarrow$) Suppose that G_1, G_2 are open. We prove that $f(x)$ is continuous at all $x_0 \in \mathbb{R}$. Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$ be an arbitrary positive number. Let us note that

$$(x_0, f(x_0) - \epsilon) \in G_1, \text{ and } (x_0, f(x_0) + \epsilon) \in G_2.$$

Furthermore, G_1 and G_2 are open sets. We can find sufficiently small $\delta_0 > 0$ satisfying both

$$B((x_0, f(x_0) - \epsilon), \delta_0) \subset G_1 \text{ and } B((x_0, f(x_0) + \epsilon), \delta_0) \subset G_2$$

Now let us pick arbitrary $x \in B(x_0, \delta_0)$. Since $|x - x_0| < \delta_0$, note that

$$(x, f(x_0) - \epsilon) \in B((x_0, f(x_0) - \epsilon), \delta_0) \subset G_1,$$

so $(x, f(x_0) - \epsilon) \in G_1$. Therefore, $f(x_0) - \epsilon < f(x) \Leftrightarrow -\epsilon < f(x) - f(x_0)$. Similarly,

$$(x, f(x_0) + \epsilon) \in B((x_0, f(x_0) + \epsilon), \delta_0) \subset G_2.$$

so $(x, f(x_0) + \epsilon) \in G_2$. Therefore, $f(x_0) + \epsilon > f(x) \Leftrightarrow f(x) - f(x_0) < \epsilon$. Now we conclude that $\forall x \in B(x_0, \delta_0)$, $|f(x) - f(x_0)| < \epsilon$, hence $f(x)$ is continuous at x_0.

$\square$

79 (Exercise 27) We prove the contraposition. We show that if $\bigcap_{\alpha\in I} F_\alpha = \emptyset$, then there exists a finite number of $\{\alpha_1, \alpha_2, \cdots, \alpha_m\} \subset I$ s.t $\bigcap_{i=1}^m F_{\alpha_i} = \emptyset$.

Let $G_\alpha \overset{\text{def}}{=} F_\alpha^c = \mathbb{R}^d \setminus F$. Note that

$$\bigcup_{\alpha\in I} G_\alpha = \mathbb{R}^d.$$

We arbitrarily pick $\alpha_0 \in I$. Then $F_{\alpha_0} \subset \bigcup_{\alpha\in I} G_\alpha$. G_α is open for all $\alpha \in I$. By Heine-Borel's Covering Theorem, we can find a finite number of $\{\alpha_1, \cdots, \alpha_m\} \subset I$ s.t

$$F_{\alpha_0} \subset \bigcup_{i=1}^m G_{\alpha_i}.$$

By taking the complement of the both sides, we have

$$\bigcap_{i=1}^m F_{\alpha_i} \subset F_{\alpha_0}^c.$$

Therefore,

$$\bigcap_{i=0}^m F_{\alpha_i} \subset F_{\alpha_0} \cap F_{\alpha_0}^c = \emptyset.$$

Now the proof is complete. □

80 (Exercise 28) Let $F \overset{\text{def}}{=} G^c = \mathbb{R}^d \setminus G$. F is closed. Let $F_\alpha^* = F_\alpha \cap F$, then F_α^* is also bounded closed. Note that

$$\begin{aligned} \emptyset = \left(\bigcap_{\alpha\in I} F_\alpha\right) \cap G^c &= \left(\bigcap_{\alpha\in I} F_\alpha\right) \cap F \\ &= \bigcap_{\alpha\in I} (F_\alpha \cap F) \\ &= \bigcap_{\alpha\in I} F_\alpha^*. \end{aligned}$$

By the conclusion of Exercise 27 (contraposition of the original statement), we can find a finite number of $\{\alpha_1, \cdots, \alpha_m\} \subset I$ s.t

$$\bigcap_{i=1}^m F_{\alpha_i}^* = \emptyset.$$

So

$$\begin{aligned} \bigcap_{i=1}^m F_{\alpha_i}^* &= \bigcap_{i=1}^m (F_{\alpha_i} \cap F) \\ &= \left(\bigcap_{i=1}^m F_{\alpha_i}\right) \cap F = \emptyset. \end{aligned}$$

This implies that $\bigcap_{i=1}^m F_{\alpha_i}$ and $F = G^c$ are disjoint. So $\bigcap_{i=1}^m F_{\alpha_i} \subset F^c = G$. (if A and B are disjoint, then $A \subset B^c$ and $B \subset A^c$.) Now the proof is complete. □

81 (Exercise 29) We consider the negation of the statement. So we suppose that $\forall \epsilon_0 > 0$, $\exists x_0 \in K$ s.t $\forall k \geqq 1$, $B(x_0, \epsilon_0) \not\subset G_k$. Let us put $\epsilon \leftarrow \frac{1}{n}$. For each $n \in \mathbb{N}$, there exists $x_n \in K$ s.t $B(x_n, \frac{1}{n}) \not\subset G_k$ for all $k \geqq 1$. Note that $\{x_n\} \subset K$ and K is bounded and closed, we can find a subsequence $x_{n_i} \to x^* \in K$. Since $\{G_k\}_{k \geqq 1}$ covers K, and $x^* \in K$, there exists $k^* \in \mathbb{N}$ s.t $x^* \in G_{k^*}$. G_{k^*} is an open set, we can find ϵ^* s.t $B(x^*, \epsilon^*) \subset G_{k^*}$. Now let us choose sufficiently large $n \in \mathbb{N}$ s.t $|x_n - x^*| < \frac{1}{2\epsilon^*}$ and $\frac{1}{2n} < \epsilon^*$. Then

$$B\left(x_n, \frac{1}{n}\right) \subset B\left(x^*, \epsilon^*\right) \subset G_{k^*}.$$

This contradicts to the fact that for each $n \in \mathbb{N}$, $B(x_n, \frac{1}{n}) \not\subset G_k$ for all $k \geqq 1$. Now the proof is complete. □

82 (Exercise 30) The proof is the same as Exercise 19. All we have to do is to prove that $f'(x)$ has intermediate value property. It is known that if $f(x)$ is differentiable, $f'(x)$ has intermediate value property.

Suppose that $a < b$ and $f'(a) < f'(b)$ holds. (The proof for the case $f'(a) > f'(b)$ is similar.) We prove that $\forall \mu \in (f'(a), f'(b))$, there exists $c \in (a, b)$ s.t $f'(c) = \mu$. Let $F(x) \stackrel{\text{def}}{=} f(x) - \mu x, (x \in [a, b])$. Since $f(x)$ is differentiable, $F(x)$ is also differentiable. Note that $F'(a) = f'(a) - \mu < 0$ and $F'(b) = f'(b) - \mu > 0$. This implies that $F(x)$ is decreasing around a and increasing around b. Furthermore $F(x)$ is continuous on $[a, b]$, so $F(x)$ has a minimum value at some $c \in (a, b)$. Then $F'(c) = 0 = f'(c) - \mu$. Now the proof is complete. □

83 (Exercise 31) We prove that $R(f) \stackrel{\text{def}}{=} \{f(x) \mid x \in \mathbb{R}\}$ is open and closed. Then $R(f) = \emptyset$ or $\mathbb{R}$ by the conclusion of Exercise 21.

STEP 1. ($R(f)$ is closed) We show that $R(f)' \subset R(f)$. When $R(f)' = \emptyset$, $R(f)' \subset R(f)$ holds obviously, so we suppose that $R(f)' \neq \emptyset$. Let us pick $y^* \in R'(f)$. There exists $\{y_n\} \subset R(f)$ s.t $y_n \to y^* \in \mathbb{R}$. Since $y_n \in R(f)$, there exists $x_n \in \mathbb{R}$ s.t $y_n = f(x_n)$. Now by assumption,

$$|f(x_n) - f(x_m)| \geqq a|x_n - x_m|.$$

By taking $n, m \to \infty$,

$$0 \stackrel{(*)}{=} \lim_{n,m\to\infty} |f(x_n) - f(x_m)| \geqq \limsup_{n,m\to\infty} a|x_n - x_m|.$$

- (∗) $f(x_n), f(x_m) \to y^* \in \mathbb{R}$.

So $\{x_n\}_{n \geq 1}$ is a Cauchy sequence. (By completeness of real number,) a Cauchy sequence converges. Let $x_0 \stackrel{\text{def}}{=} \lim_{n\to\infty} x_n,\ (x_0 \in \mathbb{R})$. Since $f(x)$ is continuous, we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} f(x_n) = f(x_0).$$

The left hand side is y^*. So $y^* = f(x_0)$. This implies that $y^* \in R(f)$. Now the proof for this part is complete.

STEP 2. ($R(f)$ is open) Since

$$|f(x) - f(y)| \geqq a|x - y|,$$

if $f(x) = f(y)$, then $|x - y| = 0 \Leftrightarrow x = y$. So $f(x)$ is one-to-one. Since $f(x)$ is continuous and one-to-one, $f(x)$ is strictly monotone increasing (or decreasing). Let $g(x) \stackrel{\text{def}}{=} f^{-1}(x)$. Note that $R(f) \stackrel{\text{def}}{=} \{f(x) \mid x \in \mathbb{R}\} = \{g^{-1}(x) \mid x \in \mathbb{R}\} = g^{-1}(\mathbb{R})$. When $f(x)$ is strictly monotone increasing and continuous, $g(x) = f^{-1}(x)$ is also continuous. Since $\mathbb{R}$ is an open set, so $g^{-1}(\mathbb{R})$ is also open. Now the proof is complete.

STEP 3. (Supplement (I)) We prove that if $f(x) : \mathbb{R} \mapsto \mathbb{R}$ is continuous and one-to-one, then $f(x)$ is either strictly monotone increasing (or decreasing).

First, we claim that if $f(x) : \mathbb{R} \mapsto \mathbb{R}$ is continuous and one-to-one, and suppose $a < c < b$ and $f(a) < f(b)$, then $f(a) < f(c) < f(b)$. Suppose that $f(c) < f(a) < f(b)$. Let us pick $\alpha \in (f(c), f(a))$. By intermediate value theorem, there exists x, y $(a < x < c < y < b)$ s.t $\alpha = f(x) = f(y)$. However, this contradicts to the fact that $f(x)$ is one-to-one. So $f(c) < f(a)$ can not happen. Similarly, $f(a) < f(b) < f(c)$ also can not happen. So we conclude that $f(a) < f(c) < f(b)$.

By applying the same argument to $[a, c]$ and $[c, b]$, if $a < g < b < h < c$, then $f(a) < f(g) < f(c) < f(h) < f(b)$. And we conclude that $f(x)$ is strictly monotone increasing (or decreasing) on any interval $[a, b]$, so on $(-\infty, \infty)$.

STEP 4. (Supplement (II)) We prove that if $f(x) : \mathbb{R} \mapsto \mathbb{R}$ is continuous and strictly monotone increasing (or decresing), then $f^{-1}(x)$ is also continuous. Let $\epsilon > 0$ be an arbitrary positive number, and let $y_0 = f(x_0)$. We show that $\exists \delta > 0$ s.t $\forall y \in B(f(x_0), \delta)$, $|f^{-1}(y) - f^{-1} \circ f(x_0)| < \epsilon$. Since $f(x)$ is strictly monotone incresing, $f(x_0 - \epsilon) < f(x_0) < f(x_0 + \epsilon)$. Let $\delta > 0$ with $\delta < \min\{f(x_0 + \epsilon) - f(x_0), f(x_0) - f(x_0 - \epsilon)\}$. Then we have

$$f(x_0 - \epsilon) < f(x_0) - \delta < f(x_0) + \delta < f(x_0 + \epsilon).$$

If $y \in (f(x_0) - \delta, f(x_0) + \delta) = B(f(x_0), \delta)$, then $f^{-1}(y) \in (x_0 - \epsilon, x_0 + \epsilon)$, because $f(x)$ is strictly monotone increasing. So $|f^{-1}(y) - x_0| = |f^{-1}(y) - f^{-1} \circ f(x_0)| < \epsilon$. Now the proof is complete.

□

84 (Exercise 32) The proof is quite similar to Example 13 ($E = \mathbb{Q}$). Suppose that $E = \{e_n\}_{n \geqq 1}$ is a G_δ set. Then there exists a countable number of open sets $\{G_n\}_{n \geqq 1}$ s.t

$$E = \bigcap_{n=1}^{\infty} G_n.$$

Since $E \subset G_n$, G_n is also dense in $\mathbb{R}$. Let $F_n = G_n^c$. F_n is a closed set and F_n has no interior point $(*)$. Finally,

$$\mathbb{R} = (\mathbb{R} \setminus E) \cup E = \bigcup_{n=1}^{\infty} F_n \cup \bigcup_{n=1}^{\infty} \{e_n\},$$

so $\mathbb{R}$ is a countable union of closed sets with no interior point. By Baire's theorem, $\mathbb{R}$ has no interior point.(contradiction!!)

$(*)$ We prove that if G is dense then $F = G^c$ has no interior point. Suppose that F has an interior point, then $\exists x_0 \in F$ and $\exists \delta_0 > 0$ s.t $B(x_0, \delta_0) \subset F$. Since G is dense, there exists a sequence $\{x_n\} \subset G$ s.t $x_n \to x_0$. However, when n is large enough, $|x_n - x_0| < \delta_0$, so $x_n \in B(x_0, \delta_0) \subset F$, and this contradicts to the assumption that $x_n \in G$. □

85 (Exercise 34) Let us recall that the set of points of continuity of $f(x)$ is a G_δ set. (See Example 11.) And we also show that $\mathbb{Q}$ is not a G_δ set. (See Example 13.) From these two facts, it follows that $f(x)$ can not be continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{R} \setminus \mathbb{Q}$. Now the proof is complete. □

86 (Exercise 37) We show that every closed set F on $\mathbb{R}^d$ is a G_δ set. Let

$$f(x) \overset{\text{def}}{=} \operatorname{dist}(x, F).$$

We claim that

$$F = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, F) = 0\}.$$

First, $\subset$ is obviously holds. Second, let us recall that if F is a non-empty closed set, then for all $x \in \mathbb{R}^d$, there exists $y \in F$ s.t $|x - y| = \operatorname{dist}(x, F)$. (See Theorem 1.24.) So $\operatorname{dist}(x, F) = 0$ implies that $|x - y| = 0$ for some $y \in F$, hence $x = y \in F$. Now the proof for the claim above is complete.

Since

$$\{x \in \mathbb{R}^d \mid \operatorname{dist}(x, F) = 0\} = \bigcap_{n=1}^{\infty} \left\{x \in \mathbb{R}^d \mid \operatorname{dist}(x, F) < \frac{1}{n}\right\},$$

and $\operatorname{dist}(x, F)$ is (uniformly) continuous (Theorem 1.25), so $\left\{x \in \mathbb{R}^d \mid \operatorname{dist}(x, F) < \frac{1}{n}\right\}$ is an open set on $\mathbb{R}^d$ for each $n \in \mathbb{N}$, hence the right hand side is a G_δ set. Now the proof is complete. □

87 (Exercise 38)

STEP 1. Let $\{a_n\}_{n \geqq 1}$ a sequence. First we explain that we can find a subsequence n_k s.t $a_{n_k} \to \limsup_{n\to\infty} a_n$. Let $\overline{a} \overset{\text{def}}{=} \limsup_{n\to\infty} a_n$. Let us recall that

$$\overline{a} = \limsup_{n\to\infty} a_n \overset{\text{def}}{=} \lim_{n\to\infty} \sup_{m \geqq n} a_m.$$

Let $b_n \overset{\text{def}}{=} \sup_{m \geqq n} a_m$, then $b_n \searrow \overline{a}$. Since $b_n = \sup_{m \geqq n} a_m$, we can find a subsequence n_k s.t $a_{n_k} \leqq b_k \leqq a_{n_k} + \frac{1}{k}$. Finally, $0 \leqq b_k - a_{n_k} \leqq \frac{1}{k} \to 0$ and $b_k \to \overline{a}$, so $a_{n_k} \to \overline{a}$.

STEP 2. Let us pick $x_0 \in [0, 1]$ and let us consider an arbitrary sequence $\{x_n\} \subset [0, 1]$ s.t $x_n \to x_0$. Note that $\{(x_n, f(x_n))\} \subset G_f$. Let us pick a subsequence x_{n_k} s.t $f(x_{n_k}) \to \limsup_{n\to\infty} f(x_n) < \infty$. ($< \infty$ holds because G_f is bounded.) Since G_f is closed, $(x_{n_k}, f(x_{n_k})) \to (x_0, \limsup_{n\to\infty} f(x_n)) \in G_f$. This implies that $f(x_0) = \limsup_{n\to\infty} f(x_n)$. By repeating a similar argument, we also have $f(x_0) = \liminf_{n\to\infty} f(x_n)$.

□

88 (Exercise 39) We prove the contraposition. Suppose that F is not closed, and we prove that there exists a continuous function $f(x) \in C(F)$ which has no continuous extension. Since $F' \not\subset F$, $F' \setminus F \neq \emptyset$. We can pick $x_0 \in F' \setminus F$. Let us define

$$f(x) \overset{\text{def}}{=} \frac{1}{|x - x_0|}, \ (x \in F)$$

Obviously, $f(x)$ is continuous on F. (Note that $x_0 \notin F$.) Suppose that there exists $g(x) \in C(\mathbb{R})$ with $f(x) = g(x)$ for all $x \in F$. Let us pick $\{x_n\} \subset F$ s.t $x_n \to x_0$. $(x_0 \in F')$. Then

$$g(x_0) \overset{(*1)}{=} \lim_{n\to\infty} g(x_n) \overset{(*2)}{=} \lim_{n\to\infty} f(x_n) = \infty$$

- $(*1)$ $g(x)$ is continuous on $\mathbb{R}$.
- $(*2)$ $g(x) = f(x)$ on F and note that $x_n \in F$ for all $n \geqq 1$.

This implies that $f(x)$ has no continuous extension on $\mathbb{R}$. □

CHAPTER 2

Solutions

§ 2.1

1 (Definition 2.1) We define $m^*(E)$ as below.

$$\inf_{\Gamma}\left\{\sum_{I\in\Gamma}|I| \mid E\subset\bigcup_{I\in\Gamma} I,\ \Gamma \text{ is a collection of at most a countable number of open rectangles.}\right\}$$

- Note that in the definition above, Γ is a collection of at most a countable number of open rectangles, so we also allow Γ to be a collection of a finite number of open rectangles.
- Note that $m^*(E) \geqq 0$ holds obviously for any $E \subset \mathbb{R}^d$.

□

2 (Example 1) This problem claims that a set which consists of a single point has measure zero. Let

$$I_n \overset{\text{def}}{=} \begin{cases} \prod_{i=1}^{d}\left(x_0-\frac{\epsilon}{2}, x_0+\frac{\epsilon}{2}\right) & n=1 \\ \emptyset & n \geqq 2 \end{cases}.$$

Note that

$$\{x_0\} \subset \bigcup_{n=1}^{\infty} I_n,$$

and $\{I_n\}_{n\geqq 1}$ is a collection of a countable number of open rectangles. So by the definition of $m^*(\{x_0\})$, we have

$$0 \leqq m^*(\{x_0\}) \overset{(*)}{\leqq} \sum_{n=1}^{\infty}|I_n| = |I_1| = \prod_{i=1}^{d}\epsilon = \epsilon^d.$$

(*) holds because according to the definition (See Definition 2.1) of $m^*(\{x_0\})$, we take infimum of $\sum_{I\in\Gamma}|I|$ with respect to Γ, so $m^*(\{x_0\})$ is less than or equal to $\sum_{I\in\Gamma}|I|$ for any Γ which is a cover of $\{x_0\}$. Since $\epsilon > 0$ is an arbitrary positive number, we have the desired result by taking $\epsilon \to 0$. $\square$

3 (Example 2)

(1)

STEP 1. Let $\epsilon > 0$ be an arbitrary positive number and let

$$J \overset{\text{def}}{=} \prod_{i=1}^{d}\left(a_i - \frac{\epsilon}{2}, b_i + \frac{\epsilon}{2}\right).$$

Note that $\overline{I} \subset J$. Let $\Gamma \overset{\text{def}}{=} \{J\}$, then Γ is a finite cover of I. By Definition 2.1,

$$m^*(\overline{I}) \leq |J| = \prod_{i=1}^{d}(b_i - a_i + \epsilon).$$

Since $\epsilon > 0$ is an arbitrary positive number, by taking $\epsilon \to 0$, we have

$$m^*(\overline{I}) \leqq \prod_{i=1}^{d}(b_i - a_i) = |I|.$$

STEP 2. Let us consider an open cover of $\overline{I} \subset \bigcup_{n=1}^{\infty} I_n$ Since $\overline{I}$ is bounded and closed, we can find a finite subcover. (Theorem 1.21 Heine-Borel's Covering Theorem.) So $I \subset \overline{I} \subset \bigcup_{k=1}^{K} I_{n_k}$. Since the number of open rectangles which cover I is finite, we have

$$|I| \overset{(*1)}{\leqq} \sum_{k=1}^{K}|I_{n_k}| \overset{(*2)}{\leqq} \sum_{n=1}^{\infty}|I_n|.$$

Finally let us take infimum of the right hand side with respect to an open cover $\{I_n\}_{n\geqq 1}^{\infty}$. By Definition 2.1, we have

$$|I| \leqq m^*(\overline{I}).$$

- (*1) As we have stated in the question part, we suppose that if $I \subset \bigcup_{n=1}^{k} I_n$ (I, I_n: open rectangles, k is finite.), then $|I| \leqq \sum_{n=1}^{k}|I_k|$.
- (*2) This holds obviously.

Someone may feel that this solution is roundabout (or doing something unnecessary). However, when $I \subset \bigcup_{n=1}^{\infty} I_n$, we can not directly conclude that $|I| \leqq \sum_{n=1}^{\infty}|I_n|$. So we first need to find a finite cover of $\overline{I}$.

(2) The solutions is similar to the previous case.

STEP 1. Similarly let $J \overset{\text{def}}{=} \prod_{i=1}^{d}\left(a_i - \frac{\epsilon}{2}, b_i + \frac{\epsilon}{2}\right)$. Then $I \subset J$ and let $\Gamma \overset{\text{def}}{=} \{J\}$. Since Γ is a finite cover of I, we have $m^*(I) \leqq |J| = \prod_{i=1}^{d}(b_i - a_i + \epsilon)$. By taking $\epsilon \to 0$, we have $m^*(I) \leqq \prod_{i=1}^{d}(b_i - a_i) = |I|$.

STEP 2. Similarly consider the cover of I. Suppose that $I \subset \bigcup_{n=1}^{\infty} I_n$. Let $I_\epsilon = \prod_{i=1}^{d}\left(a_i + \frac{\epsilon}{2}, b_i - \frac{\epsilon}{2}\right)$. Note that $I_\epsilon \subset \overline{I}_\epsilon \subset I \subset \bigcup_{n=1}^{\infty} I_n$. Since $\overline{I}_\epsilon$ is bounded and closed, and $\overline{I}_\epsilon \subset \bigcup_{n=1}^{\infty} I_n$, we can find a finite subcover s.t

$$\overline{I}_\epsilon \subset \bigcup_{k=1}^{K} I_{n_k}.$$

Since

$$I_\epsilon \subset \bigcup_{k=1}^{K} I_{n_k},$$

we have

$$|I_\epsilon| \leqq \sum_{k=1}^{K} |I_{n_k}| \leqq \sum_{n=1}^{\infty} |I_n|.$$

By taking infimum with respect to $\{I_n\}_{n \geqq 1}$ on the right hand side, we have

$$|I_\epsilon| \leqq m^*(I).$$

Note that the left hand side is

$$|I_\epsilon| = \prod_{i=1}^{d}(b_i - a_i - \epsilon).$$

Finally, by taking $\epsilon \to 0$, we have $|I_\epsilon| \to |I|$, hence

$$|I| \leqq m^*(I).$$

□

4 (Theorem 2.1)

(1) Suppose $E \subset \bigcup_{n \geqq 1} I_n$. For all open covers, $\sum_{n \geqq 1} |I_n| \geqq 0$. So $m^*(E) \geqq 0$. Let I with $|I| < \epsilon$. $\forall I, \emptyset \subset I$. So $m^*(I) \leqq |I| < \epsilon$

(2) Let us consider an open cover of B, Γ_B. Let $\Gamma_B \stackrel{\text{def}}{=} \{I_n^{(B)}\}_{n=1}^{\infty}$. $B \subset \bigcup_{n=1}^{\infty} I_n^{(B)}$. Of course, $A \subset \bigcup_{n \geqq 1} I_n^{(B)}$. So $m^*(A) \leqq \sum_{n=1}^{\infty} \left|I_n^{(B)}\right|$ for any Γ_B. Take infimum of the right hand side with respect to Γ_B. Then we have $m^*(A) \leqq m^*(B)$.

(3) For each $n = 1, 2 \cdots$, suppose $A_n \subset \bigcup_{m \geqq 1} I_{n,m}$ with $m^*(A_n) \leqq \sum_{m \geqq 1} |I_{n,m}| < m^*(A_n) + \frac{\epsilon}{2^n}$. Since $\bigcup_{n \geqq 1} \bigcup_{m \geqq 1} I_{n,m}$ is an open cover of $\bigcup_{n \geqq 1} A_n$, we have $m^*(\bigcup_{n=1} A_n) \leqq \sum_{n=1} \sum_{m=1} |I_{n,m}| < \sum_{n=1}^{\infty} \left(m^*(A_n) + \frac{\epsilon}{2^n}\right) = \sum_{n=1} m^*(A_n) + \epsilon$. Finally by taking $\epsilon \searrow 0$, we have the desired result.

□

5 (Corollary 2.2) We present a proof in the case of $d = 1$. (Extension to the general case is easy.) Suppose $E \overset{\text{def}}{=} \{x_k\}_{k=1}^{\infty}$. Let us consider $I_k \overset{\text{def}}{=} \left(x_k - \frac{\epsilon}{2^{k+1}}, x_k + \frac{\epsilon}{2^{k+1}}\right)$. Let us pay attention to the fact that $E \subset \bigcup_{k=1}^{\infty} I_k$. Then we have $m^*(E) \leqq \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$ by the definition of outer measure. This implies that $m^*(E) = 0$. □

6 (Lemma 2.3) We show $m^*(E) \leqq m^*_\delta(E)$ and $m^*(E) \geqq m^*_\delta(E)$.

(1) $m^*(E) \leqq m^*_\delta(E)$ holds obviously from their definitions because $A \subset B \Rightarrow \inf A \geqq \inf B$ holds. (Let $\{I_n\}_{n=1}^{\infty}$ be a cover of E and suppose that the edge length of each $I_n < \delta$. In the definition of $m^*(E)$, we also consider such $\{I_n\}$ because it also covers E. So $m^*(E) \leqq \sum_{k=1}^{\infty} |I_k|$. By taking the infimum of the right hand side, we have $m^*(E) \leqq m^*_\delta(E)$.)

(2)

STEP 1. Let us consider a cover of E. Suppose that $E \subset \bigcup_{n=1}^{\infty} I_n$ (I_n are open rectangles). For each I_n, we devide I_n into smaller disjoint open rectangles $\{I_{n,k}\}(k = 1 \cdots K_n)$ whose edge length is all less than $\frac{\delta}{2}$. (You can easily imagine that you can do so when $d = 1$. Of course so is $d > 1$.)

However $\{I_{n,k}\}(k = 1 \cdots K_n)$ does not actually cover I_n because the boundary points are lost. So we enlarge each $I_{n,k}$ by $\lambda \in (1, 2)$ times without moving its center so that $\{\lambda I_{n,k}\}_{k=1}^{K_n}$ will cover I_n. Now we have $E \subset \bigcup_{n=1} I_n \subset \bigcup_{n=1} \bigcup_{k=1}^{K_n} \lambda I_{n,k}$.

STEP 2. From the fact that $E \subset \bigcup_{n=1} \bigcup_{k=1}^{K_n} \lambda I_{n,k}$, we have

$$m^*_\delta(E) \leqq \sum_{n=1}^{\infty} \sum_{k=1}^{K_n} |\lambda I_{n,k}| = |\lambda|^d| \sum_{n=1}^{\infty} \sum_{k=1}^{K_n} |I_{n,k}|.$$

Let us be careful of the fact that $|I_n| = \sum_{k=1}^{K_n} |I_{n,k}|$. (If you do not know why, let us consider a simpler case. $|(a, b)| + |(b, c)| = (b - a) + (c - b) = c - a = |(a, c)|$.) Therefore,

$$|\lambda|^d| \sum_{n=1}^{\infty} \sum_{k=1}^{K_n} |I_{n,k}| = |\lambda|^d| \sum_{n=1}^{\infty} |I_n|.$$

By taking the infimum with respect to $\{I_n\}_{n=1}^{\infty}$, we have $m^*_\delta(E) \leqq |\lambda^d| m^*(E)$. The argument above holds for all $\lambda \in (1, 2)$. Finally by taking $\lambda \searrow 1$, we have the desired conclusion.

□

7 (Theorem 2.4)

(1)

STEP 1. By Theorem 2.1 (and Lemma 2.3), we have

$$m^*_\delta(E_1 \cup E_2) \leqq m^*_\delta(E_1) + m^*_\delta(E_2).$$

STEP 2. Next we prove that

$$m^*_\delta(E_1) + m^*_\delta(E_2) \leqq m^*_\delta(E_1 \cup E_2).$$

Let $\delta \overset{\text{def}}{=} \frac{1}{\sqrt{d}}\text{dist}(E_1, E_2) > 0$. If the edge length of I_n is less than δ, then we have

$$\text{diam}(I_n) < \left(\sum_{i=1}^{d}\left(\frac{1}{\sqrt{d}}\text{dist}(E_1, E_2)\right)^2\right)^{1/2} = \text{dist}(E_1, E_2).$$

Let us consider a cover of $E_1 \cup E_2$ by a countable number of open rectangles $\{I_k\}_{k\geqq 1}$ whose edge length is less than δ. (i.e $E_1 \cup E_2 \subset \bigcup_{k=1}^{\infty} I_k$.) Without loss of generality, we may suppose $(E_1 \cup E_2) \cap I_k \neq \emptyset$ for every $k \in \mathbb{N}$. (If $E_1 \cup E_2 \cap I_k = \emptyset$, we may get rid of it from the cover.)

Note that $E_1 \cap I_k = \emptyset$ or $E_2 \cap I_k = \emptyset$, and I_n can not have a common point with both E_1 and E_2 simultaneously because $\text{diam}(I_n)$ is less than $\text{dist}(E_1, E_2)$. So we can always separate $\{I_k\}_{k\geqq 1}$ into $\left\{I_k^{(1)}\right\}_{k\geqq 1} \cup \left\{I_k^{(2)}\right\}_{k\geqq 1}$ where $E_1 \subset \bigcup_{k=1}^{\infty} I_k^{(1)}$ and $E_2 \subset \bigcup_{k=1}^{\infty} I_k^{(2)}$. Also note that

$$m_\delta^*(E_1) \leqq \sum_{k=1}^{\infty}\left|I_k^{(1)}\right|,\ m_\delta^*(E_2) \leqq \sum_{k=1}^{\infty}\left|I_k^{(2)}\right|,$$

hence,

$$m_\delta^*(E_1) + m_\delta^*(E_2) \leqq \sum_{i=1}^{2}\sum_{k=1}^{\infty}\left|I_k^{(i)}\right| = \sum_{n=1}^{\infty}|I_n|.$$

Finally, by taking infimum on the right hand side with respect to $\{I_n\}_{n\geqq 1}$, we have

$$m_\delta^*(E_1) + m_\delta^*(E_2) \leqq m_\delta^*(E_1 \cup E_2).$$

(2) Since

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leqq \sum_{n=1}^{\infty} m^*(E_n)$$

holds by sub-additivity of outer measure (Theorem 2.1), it is enough for us to prove that

$$\sum_{n=1}^{\infty} m^*(E_n) \leqq m^*\left(\bigcup_{n=1}^{\infty} E_n\right).$$

For each $N \in \mathbb{N}$, $\bigcup_{n=1}^{N} E_n \subset \bigcup_{n=1}^{\infty} E_n$, so we have

$$m^*\left(\bigcup_{n=1}^{N} E_n\right) \leqq m^*\left(\bigcup_{n=1}^{\infty} E_n\right),$$

by monotonicity of outer measure (Theorem 2.1). We claim that the left hand side

$$m^*\left(\bigcup_{n=1}^{N} E_n\right) = \sum_{n=1}^{N} m^*(E_n).$$

If this holds, then

$$\sum_{n=1}^{N} m^*(E_n) \leqq m^*\left(\bigcup_{n=1}^{\infty} E_n\right)$$

for all $N \in \mathbb{N}$. By taking $N \to \infty$, we have the desired result. Now we prove the claim above. First we show that $\text{dist}(E_1, \bigcup_{n=2}^{N} E_n) > 0$. If this is true, then

$$m^*\left(\bigcup_{n=1}^{N} E_n\right) = m^*\left(E_1 \cup \bigcup_{n=2}^{N} E_n\right) = m^*(E_1) + m^*\left(\bigcup_{n=2}^{N} E_n\right),$$

from the previous result. By repeating the similar argument, we have $m^*\left(\bigcup_{n=1}^{N} E_n\right) = \sum_{n=1}^{N} m^*(E_n)$. So all we have to do is to prove that

$$\text{dist}(E_1, \bigcup_{n=2}^{N} E_n) > 0.$$

By definition,

$$\begin{aligned} \text{dist}(E_1, \bigcup_{n=2}^{N} E_n) &= \inf_{x \in E_1, y \in \bigcup_{n=2}^{N} E_n} |x - y| \\ &\overset{(*)}{\geqq} \min_{n=2,\cdots,N} \inf_{x \in E_1, y \in E_n} |x - y| \\ &= \min_{n=2,\cdots,N} \text{dist}(E_1, E_n) > 0 \end{aligned}$$

Finally, we explain $(*)$. By the definition of infimum, we can find a sequence $\{x_k\} \subset E_1, \{y_k\} \subset \bigcup_{n=2}^{N} E_n$ s.t

$$|x_k - y_k| \searrow \inf_{x \in E_1, y \in \bigcup_{n=2}^{N} E_n} |x - y|$$

And there exists some $n_0 \in \{2, \cdots, N\}$ s.t $y_k \in E_{n_0}$ for infinitely many k. So we can find a subsequence k_ℓ s.t $y_{k_\ell} \in E_{n_0}$. Finally,

$$\begin{aligned} \lim_{k \to \infty} |x_k - y_k| &= \lim_{\ell \to \infty} |x_{k_\ell} - y_{k_\ell}| \\ &\geqq \inf_{x \in E_1, y \in E_{n_0}} |x - y| \\ &\geqq \min_{n=2,\cdots,N} \inf_{x \in E_1, y \in E_n} |x - y|. \end{aligned}$$

Now the proof is complete.

□

8 (Theorem 2.5 (a)) Suppose $E \subset \bigcup_{n=1}^{\infty} I_n$. Then $E_{+x_0} \subset \bigcup_{n=1}^{\infty} I_{n+x_0}$. $m^*(E_{+x_0}) \leqq \sum_{n=1}^{\infty} |I_{n+x_0}| = \sum_{n=1}^{\infty} |I_n|$. Finally let us take infimum of the right hand side. In the same way we may prove $\geqq$. □

9 (Theorem 2.5 (b)) $E \subset \bigcup_{n=1}^{\infty} I_n$. Then $\lambda E \subset \bigcup_{n=1}^{\infty} \lambda I_n$. So we have $m^*(\lambda E) \leqq \sum_{n=1}^{\infty} |\lambda|^d |I_n|$. By taking infimum, we have $m^*(\lambda E) \leqq |\lambda|^d m^*(E)$.

This holds even if we change λ to $\frac{1}{\lambda}$. So we have $m^*(\frac{1}{\lambda}E) \leqq \frac{1}{|\lambda|^d} m^*(E)$ We can also change E to λE. Then $m^*(E) \leqq \frac{1}{|\lambda|^d} m^*(\lambda E)$. Now we have the desired conclusion. □

10 (Generalized definition of outer measure) $\mu^*: 2^X \to [0,\infty]$ is an outer measure when it satisfies the following conditions.

(3) (non-negtative) $\forall A \subset X, \mu^*(A) \geqq 0$ and $\mu^X(\emptyset) = 0$.

(4) (monotone) If $A \subset B(\subset X)$, $\mu^*(A) \leqq \mu^*(B)$.

(5) (countable sub-additive)Let $A_n \subset X$ for all $n \geqq 1$. Then $\mu^*\left(\bigcup_{n\geqq 1} A_n\right) \leqq \sum_{n\geqq 1}\mu^*(A_n)$.

□

11 (Exercise 1) Note that $A \cup B = A \cup (B \setminus A) = B \cup (A \setminus B)$.

$$\begin{aligned} m^*(B) \overset{*1}{\leqq} m^*(A\cup B) &= m^*(A \cup (B\setminus A)) \\ &\overset{*2}{\leqq} m^*(A) + m^*(B \setminus A) \\ &\overset{*3}{=} m^*(B\setminus A) \\ &\overset{*4}{\leqq} m^*(B). \end{aligned}$$

- $(*1)$ $B \subset A \cup B$. Theorem 2.1: $m^*(\cdot)$ is monotone.
- $(*2)$ Theorem 2.1: sub-additivity
- $(*3)$ $m^*(A) = 0$.
- $(*4)$ $B \setminus A \subset B$. Theorem 2.1: $m^*(\cdot)$ is monotone.

Similarly,

$$\begin{aligned} m^*(B) \leqq m^*(A\cup B) &= m^*(B \cup (A\setminus B)) \\ &\leqq m^*(B) + m^*(A \setminus B) \\ &\overset{*5}{=} m^*(B). \end{aligned}$$

- $(*5)$ $m^*(A) = 0$ and $A \setminus B \subset A$. So $m^*(A\setminus B) = 0$.

□

12 (Exercise 2) By sub additivity and monotonicity, we have

$$\begin{aligned} m^*(A) &= m^*(A\setminus B \cup A \cap B) \\ &\leqq m^*(A\setminus B) + m^*(A\cap B) \\ &\overset{(*)}{\leqq} m^*(A\Delta B) + m^*(B). \end{aligned}$$

- $(*)$ $A \setminus B \subset A\Delta B$, $A \cap B \subset B$.

So $m^*(A) - m^*(B) \leqq m^*(A\Delta B)$. Swap A, B we have $|m^*(A) - m^*(B)| \leqq m^*(A\Delta B)$.

□

13 (Exercise 3) $E = \bigcup_{x\in E}\{x\} \subset \bigcup_{x\in E} B(x, \delta_x)$. By Lindelof's covering theorem, we can always find a countable subcover. So $E \subset \bigcup_{n=1}^{\infty} B(x_n, \delta_{x_n})$. And $E \cap E = E \subset (=) \bigcup_{n=1}^{\infty} B(x_n, \delta_{x_n}) \cap E$. By Theorem 2,1, countable sub-additivity of Lebesgue outer measure, we have

$$m^*(E) \leqq \sum_{n=1}^{\infty} m^*\left(B(x_n, \delta_{x_n}) \cap E\right) = 0$$

□

14 (Exercise 4) $f(x) \overset{\text{def}}{=} m^*([a, x] \cap E), (x \in [a, b])$. Then $f(x)$ is a continuous function on $[a, b]$. First $f(x)$ is monotone increasing on $[a, b]$. Next,

$$\begin{aligned} f(x+h) &= m^*([a, x+h] \cap E) \\ &\overset{*1}{\leqq} m^*([a, x] \cap E) + m^*([x, x+h] \cap E) \\ &\overset{*2}{\leqq} m^*([a, x] \cap E) + m^*([x, x+h]) \\ &\overset{*3}{=} m^*([a, x] \cap E) + h \\ &= f(x) + h. \end{aligned}$$

- (∗1) Theorem 2.1, sub-additivity
- (∗2) Theorem 2.1, monotonicity
- (∗3) Example 2

So $0 \leqq f(x+h) - f(x) \leqq h$. This implies that f is continuous. Finally we may prove the statement by intermediate value theorem. □

15 (Exercise 5) Let $C \subset [0, 1]$ be a Cantor set constructed in Chapter 1. Let us recall that $C = \bigcap_{n=1}^{\infty} C_n \subset C_n$. $C_n = \bigcup_{k=1}^{2^n} \overline{I}_{n,k}$. So by Theorem 2.1 (monotonicity and sub-additivity) and also by Example 2, we have

$$m^*(C) \leqq m^*(C_n) \leqq \sum_{k=1}^{2^n} m^*(\overline{I}_{n,k}) = \left(\frac{2}{3}\right)^n.$$

Finally by taking $n \nearrow \infty$, we have the desired conclusion.

□

§ 2.2

16 (Definition 2.2) Let $E \subset \mathbb{R}^d$. If the following inequality holds for all $B \subset \mathbb{R}^d$, we call that E is Lebesgue measurable. Let $\mathscr{M}$ be a collection of Lebesgue measurable sets on $\mathbb{R}^d$. (In the inquality below, $\leqq$ always holds by sub additivity of an outer measure. So we may use $=$ instead of $\geqq$.)

$$m^*(B) \geqq m^*(B \cap E) + m^*(B \cap E^c).$$

□

17 (Example 1) We show that for all $N \subset \mathbb{R}^d : m^*(N) = 0$, $N \in \mathscr{M}$. By monotonicity of an outer measure, we have

$$m^*(B \cap N) + m^*(B \cap N^c) \leqq m^*(N) + m^*(B).$$

In the inequality above, $m^*(N) = 0$, so we have the desired result. □

18 (Theorem 2.6)

(1) Since $m^*(\emptyset) = 0$, $\emptyset \in \mathscr{M}$. (See Example 1.)

(2) If $E \in \mathscr{M}$, for all $B \subset \mathbb{R}^d$,

$$m^*(B \cap E) + m^*(B \cap E^c) \leqq m^*(B).$$

So

$$m^*(B \cap E^c) + m^*(B \cap (E^c)^c) \leqq m^*(B).$$

Hence $E^c \in \mathscr{M}$

(3)

STEP 1. ($E_1 \cup E_2 \in \mathscr{M}$) Let $B, C \subset \mathbb{R}^d$ be an arbitrary subset of $\mathbb{R}^d$. E_1 is Lebesgue measurable, so we have

$$m^*(E_1 \cap C) + m^*(E_1^c \cap C) \leqq m^*(C).$$

Since C is arbitrary, so we may change $C \to B \cap E_2^c$. So we have

$$m^*(B \cap E_1 \setminus E_2) + m^*(B \cap (E_1 \cup E_2)^c) \leqq m^*(B \cap E_2^c).$$

Recall that E_2 is also measurable, so

$$m^*(B \cap E_2) + m^*(B \cap E_2^c) \leqq m^*(B).$$

So we have

$$m^*(B \cap E_1 \setminus E_2) + m^*(B \cap E_2) + m^*(B \cap (E_1 \cup E_2)^c) \leqq m^*(B)$$

Finally, by sub additivity of an outer measure,

$$m^*(B \cap (E_1 \cup E_2)) \leqq m^*(B \cap E_1 \setminus E_2) + m^*(B \cap E_2).$$

And we have the desired result.

STEP 2. ($E_1 \cap E_2, E_1 \setminus E_2$) The rest is easy. Recall that $E_1^c, E_2^c \in \mathscr{M}$. $E_1 \cap E_2 = (E_1^c \cup E_2^c)^c \in \mathscr{M}$. $E_1 \setminus E_2 = E_1 \cap E_2^c \in \mathscr{M}$.

(4)

STEP 1. Let $A_1 \stackrel{\text{def}}{=} E_1,\ A_2 \stackrel{\text{def}}{=} E_2 \setminus E_1,\ A_3 \stackrel{\text{def}}{=} E_3 \setminus (E_1 \cup E_2) \cdots$. $\{A_n\}_{n \geqq 1}$ are disjoint and $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} E_n$ and $\cup_{n=1}^{N} A_n = \bigcup_{n=1}^{N} E_n$ for all $N \in \mathbb{N}$. By the previous result $\bigcup_{n=1}^{N-1} A_n \in \mathscr{M}$. So we have, for all $C \subset \mathbb{R}^d$,

$$m^*(C) \geqq m^* \left(\bigcup_{n=1}^{N-1} A_n \cap C \right) + m^* \left(\left(\bigcup_{n=1}^{N-1} A_n \right)^c \cap C \right).$$

Since C is arbitrary, we may change $C \to B \cap \bigcup_{n=1}^{N} A_n$ where $B \subset \mathbb{R}^d$ is also arbitrary. So we have

$$m^* \left(B \cap \bigcup_{n=1}^{N} A_n \right) \geqq m^*(B \cap A_N) + m^* \left(B \cap \bigcup_{n=1}^{N-1} A_n \right).$$

By reating the similar argument, $(\bigcup_{n=1}^{N-2} A_n \in \mathscr{M})$, we will have

$$m^* \left(B \cap \bigcup_{n=1}^{N} A_n \right) \geqq \sum_{n=1}^{N} m^* \left(B \cap A_n \right).$$

STEP 2. Since $\bigcup_{n=1}^{N} A_n \in \mathscr{M}$ and by the result from the previous STEP,

$$\begin{aligned} m^*(B) &\geqq m^* \left(B \cap \bigcup_{n=1}^{N} A_n \right) + m^* \left(B \cap \left(\bigcup_{n=1}^{N} A_n \right)^c \right) \\ &\geqq \sum_{n=1}^{N} m^*(B \cap A_n) + m^* \left(B \cap \left(\bigcup_{n=1}^{N} A_n \right)^c \right) \end{aligned}$$

Moreover, $B \cap (\cup_{n=1}^{N} A_n)^c \supset B \cap (\bigcup_{n=1}^{\infty} A_n)^c$, we have

$$m^*(B) \geqq \sum_{n=1}^{N} m^*(B \cap A_n) + m^* \left(B \cap \left(\bigcup_{n=1}^{\infty} A_n \right)^c \right).$$

This holds for all $N = 1, 2 \cdots$, so we have

$$m^*(B) \geqq \sum_{n=1}^{\infty} m^*(B \cap A_n) + m^* \left(B \cap \left(\bigcup_{n=1}^{\infty} A_n \right)^c \right).$$

Finally, by sub additivity of an outer measure, we have

$$\begin{aligned} m^*(B) &\geqq \sum_{n=1}^{\infty} m^*(B \cap A_n) + m^* \left(B \cap \left(\bigcup_{n=1}^{\infty} A_n \right)^c \right) \\ &\geqq m^* \left(B \cap \bigcup_{n=1}^{\infty} A_n \right) + m^* \left(B \cap \left(\bigcup_{n=1}^{\infty} A_n \right)^c \right). \end{aligned}$$

Since $\cup_{n \geqq 1} A_n = \bigcup_{n \geqq 1} E_n$ we have $\bigcup_{n=1}^{\infty} E_n \in \mathscr{M}$.

STEP 3. If $\{E_n\}$ are disjoint, $A_n = E_n$. In the last inequality, let us consider $B \leftarrow \bigcup_{n=1}^{\infty} A_n$. And we have

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \geqq \sum_{n=1}^{\infty} m^*(A_n).$$

By sub additivity of an outer measure, we always have

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leqq \sum_{n=1}^{\infty} m^*(A_n).$$

So we have the desired conclusion.

□

19 (Theorem 2.7)

STEP 1. ($\exists k \in \mathbb{N}$ s.t $m(E_k) = \infty$) Obviously the both sides are infinite.

STEP 2. ($m(E_k) < \infty$ for all $k \in \mathbb{N}$) It is easy to verify that $A, B \in \mathscr{M}, A \subset B, m(A) < \infty$, then

$$m(B) - m(A) = m(B \setminus A).$$

First, $m(B) = m(B \setminus A \cup A) = m(B \setminus A) + m(A)$. Since $m(A) < \infty$, we can subtract $m(A)$ from the both sides. So we have $m(B) - m(A) = m(B \setminus A)$.

Let $A_k \overset{\text{def}}{=} E_k \setminus E_{k-1}, E_0 \overset{\text{def}}{=} \emptyset$.

$$\begin{aligned} m\left(\bigcup_{k=1}^{\infty} E_k\right) &= m\left(\bigcup_{k=1}^{\infty} A_k\right) \\ &\overset{*}{=} \sum_{k=1}^{\infty} m(A_k) \\ &= \lim_{k\to\infty} \sum_{m=1}^{k} m(E_k \setminus E_{k-1}) \\ &= \lim_{k\to\infty} \sum_{m=1}^{k} (m(E_k) - m(E_{k-1})) \\ &= \lim_{k\to\infty} m(E_k) \end{aligned}$$

- ($*$) Since A_k are disjoint and measurable, $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A_k)$.

□

20 (Corollary 2.8) Let $E_\infty \overset{\text{def}}{=} \bigcap_{k=1}^{\infty} E_k$. Let $A_k = E_1 \setminus E_k$. Then $A_k \nearrow E_1 \setminus E_\infty$. By the previous result $m(E_1) - m(E_\infty) = m(E_1 \setminus E_\infty) = \lim_{k\to\infty} m(A_k) = \lim_{k\to\infty}(m(E_1) - m(E_k))$. Since $m(E_1) < \infty$, so may subtract $m(E_1)$ from the both sides. □

21 (Example 2) Let $A_m = \bigcup_{k=m}^{\infty} E_k$. By sub-additivity of an outer measure, we have $m(A_1) = m\left(\bigcup_{k=1}^{\infty} E_k\right) < \infty$.

$$m\left(\bigcap_{m=1}^{\infty}\bigcup_{m=k}^{\infty} E_m\right) = m\left(\bigcap_{m=1}^{\infty} A_m\right) = \lim_{m\to\infty} m(A_m)$$

By sub-additivity,

$$= \lim_{m\to\infty} m\left(\bigcup_{k=m}^{\infty} E_k\right) \leqq \lim_{m\to\infty}\sum_{k=m}^{\infty} m\left(E_k\right) = 0.$$

Notice. Let $a_n \geqq 0$ and $\sum_{n=1}^{\infty} a_n < \infty$. Then $\lim_{k\to\infty}\sum_{n=k}^{\infty} a_n = 0$. □

22 (Corollary 2.9)

(1) Let $A_n \stackrel{\text{def}}{=} \bigcap_{m\geqq n}^{\infty} E_m$. A_n is an increasing sequence of measurable sets. So we have

$$m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{k\to\infty} m(A_k).$$

The left and side is $m(\liminf_{k\to\infty} E_k)$. Moreover,

$$m(A_k) \leqq m(E_k), \forall k \geqq 1.$$

So we have

$$\liminf_{k\to\infty} m(A_k) \leqq \liminf_{k\to\infty} m(E_k),\ \forall k \geqq 1.$$

The left hand side is $\lim_{k\to\infty} m(A_k)$ because $\lim_{k\to\infty} m(A_k)$ exists. Now the proof is complete.

(2) Let $E \stackrel{\text{def}}{=} \bigcup_{m=1}^{\infty} E_m$. Let us apply the previous result to $E_k^* \stackrel{\text{def}}{=} E \setminus E_k$.

$$m\left(\liminf_{k\to\infty} E_k^*\right) \leqq \liminf_{k\to\infty} m\left(E_k^*\right).$$

Since $m(E_k^*) = m(E \setminus E_k) = m(E) - m(E_k)$, $(\because m(E_k) < \infty)$, we can rewrite the right hand side as

$$\liminf_{k\to\infty}\left(m(E) - m\left(E_k\right)\right) = m(E) - \limsup_{k\to\infty} m(E_k).$$

Note that $\liminf_{k\to\infty} E\setminus E_k = E\setminus\limsup_{k\to\infty} E_k$, and also note that $m\left(\limsup_{k\to\infty}\right) < \infty$. Now we can rewrite the left hand side as

$$m\left(E \setminus \limsup_{k\to\infty} E_k\right) = m(E) - m\left(\limsup_{k\to\infty} E_k\right)$$

Finally since $m(E) < \infty$, so we may subtract $m(E) < \infty$ from the both sides. And we have the desired result.

□

23 (Exercise 1)

STEP 1. $(m^*(A) + m^*(B) \leqq m^*(A \cup B) + m^*(A \cap B))$ Since $A \in \mathscr{M}$, for all $B_0 \subset \mathbb{R}^n$ we have

$$m^*(B_0 \cap A) + m^*(B_0 \cap A^c) \leqq m^*(B_0).$$

Since B_0 is arbitrary, we substitute $B_0 \leftarrow A \cup B$. So we have

$$m^*(A) + m^*(B \setminus A) \leqq m^*(A \cup B).$$

By adding $m^*(A \cap B)$ to the both sides,

$$m^*(A) + m^*(B \setminus A) + m^*(A \cap B) \leqq m^*(A \cup B) + m^*(A \cap B).$$

By subadditivity, the left hand side is larger than $m^*(A) + m^*(B)$, so

$$m^*(A) + m^*(B) \leqq m^*(A \cup B) + m^*(A \cap B).$$

STEP 2. $(m^*(A \cup B) + m^*(A \cap B) \leqq m^*(A) + m^*(B))$ Since $A \in \mathscr{M}$, we have

$$m^*(A \cap B) + m^*(A^c \cap B) \leqq m^*(B).$$

By adding $m^*(A)$ to the both sides, we have

$$m^*(A \cap B) + m^*(A^c \cap B) + m^*(A) \leqq m^*(A) + m^*(B).$$

By subadditivity, $m^*(A^c \cap B) + m^*(A)$ in the left hand side is larger than $m^(A \cup B)$, so

$$m^*(A \cup B) + m^*(A \cap B) \leqq m^*(A) + m^*(B).$$

□

24 (Exercise 2) $\leqq$ always holds by sub additivity of an outer measure. In the proof of Theorem 2.6, we have already shown that for all $B \subset \mathbb{R}^d$,

$$m^*(B) \geqq \sum_{n=1}^{\infty} m^*(B \cap A_n) + m^*(B \cap (\bigcup_{n=1}^{\infty} A_n)^c),$$

so we may substitute $B \leftarrow \bigcup_{n=1}^{\infty} B_n$. Then we have the desired result. □

25 (Exercise 3) $E_1 \setminus E_2, E_2 \setminus E_1 \in \mathscr{M}$ since they are measure zero sets. We will have the desired conclusion from the formula below.

$$E_2 = E_2 \setminus E_1 \cup (E_1 \setminus (E_1 \setminus E_2)).$$

Both $m(E_1), m(E_2)$ are equal to $m(E_1 \cap E_2)$. □

26 (Exercise 4)

STEP 1.

$$\begin{aligned} m^*(\limsup E_n) &= m^*\left(\bigcap_{n=1}^{\infty}\bigcup_{k\geqq n}^{\infty} E_k\right) \\ &\leqq m^*\left(\bigcup_{k\geqq n}^{\infty} E_k\right) \\ &\leqq \sum_{k\geqq n}^{\infty} m^*(E_k). \end{aligned}$$

for all $n \in \mathbb{N}$. By taking $n \nearrow \infty$, the right hand side $\searrow 0$. So $m^*(\limsup E_n) = 0$.

STEP 2. Let

$$Z \overset{\text{def}}{=} \limsup E_n = \left\{\, x \in \mathbb{R}^1 \mid \#\{n \mid x \in E_n\} = \infty \right\}.$$

Fix $x \in \mathbb{R}^1 \setminus Z$. Then $\#\{n \,:\, x \in E_n\} < \infty$. (Only finite number of E_n contain x). Hence for sufficiently large $\forall n > N_{(x)}$, $x \notin E_n \Leftrightarrow \frac{|f_n(x)|}{\lambda_n} \leqq 1$. So $\limsup_{n\to\infty} \frac{|f_n(x)|}{\lambda_n} \leqq 1, \forall x \in \mathbb{R}^1 \setminus Z$.

□

27 (Exercise 5) Since $E \in \mathscr{M}$, for all $B \subset \mathbb{R}^d$, we have

$$m^*(T^{-1}(B)) \geqq m^*(T^{-1}(B) \cap E) + m^*(T^{-1}(B) \cap E^c).$$

Since T does not change outer measure,

$$\begin{aligned} m^*(T \circ T^{-1}(B)) &\geqq m^*(T(T^{-1}(B) \cap E)) + m^*(T(T^{-1}(B) \cap E^c)) \\ &\geqq m^*(T \circ T^{-1}(B) \cap T(E)) + m^*(T \circ T^{-1}(B) \cap T(E^c)). \end{aligned}$$

Moreover T is one-to-one and onto so $T \circ T^{-1}(B) = B$, $T(E^c) = T(E)^c$. Therefore

$$m^*(B) \geqq m^*(B \cap T(E)) + m^*(B \cap T(E)^c).$$

This implies the desired result. □

28 (Exercise 6) Let $X \overset{\text{def}}{=} \{E_\alpha\}_{\alpha\in A}$ and let

$$A_n \overset{\text{def}}{=} \left\{\alpha \in A \mid m\left(E_\alpha \cap [-n, n]\right) > \frac{1}{n}\right\}.$$

STEP 1. We prove that $A = \bigcup_{n=1}^{\infty} A_n$. Obviously $A_n \subset A$. Next, if $\alpha \in A$. Then $m(E_\alpha) > 0$. So for sufficiently large $n \in \mathbb{N}$, $m(E_\alpha \cap [-n, n]) > 0$. (Otherwise, $m(E_\alpha) = 0$ and it contradicts to the assumption.) Since $m(E_\alpha \cap [-n, n]) \nearrow m(E_\alpha) > 0$ and $\frac{1}{n} \searrow +0$, we can find $n \in \mathbb{N}$ s.t $m(E_\alpha \cap [-n, n]) > \frac{1}{n}$.

STEP 2. We show that A_n is a finite set. Since $\{E_\alpha \cap [-n,n]\}_{\alpha\in A_n}$ are also disjoint and $\bigcup_{\alpha\in A_n} E_\alpha \cap [-n,n] \subset [-n,n]$, so A_n is finite. Otherwise,

$$m\left(\bigcup_{\alpha\in A_n} E_\alpha \cap [-n,n]\right) = \sum_{\alpha\in A_n} m\left(E_\alpha \cap [-n,n]\right) > \frac{1}{n}\cdot \# A_n = \infty.$$

(But $m([-n,n]) = 2n < \infty$.)

□

29 (Exercise 7)

STEP 1. By Fatou's lemma (measure version), we have

$$m\left(\liminf_{k\to\infty} E_k\right) \leqq \liminf_{k\to\infty} m(E_k).$$

STEP 2. Since $E_k \subset [a,b]$, so $\bigcup_{k=k_0}^{\infty} E_k \subset [a,b]$ and $m\left(\bigcup_{k=k_0}^{\infty} E_k\right) \leqq b-a < \infty$, we apply Fatou's lemma (measure version), and we have

$$\limsup_{k\to\infty} m(E_k) \leqq m\left(\limsup_{k\to\infty} E_k\right).$$

(In the proof of Corollary 2.9, if we let $E = \bigcup_{k=k_0}^{\infty} E_k$ and then we have the same conclusion. So starting from $k = k_0$ does not matter because we are interested in the situation when k is sufficiently large.) Now the proof is complete.

□

30 (Exercise 8)

$$\sum_{n=1}^{\infty} \chi_{E_n}(x) < \infty,\ \forall x \in [0,1]\setminus N, m(N) = 0$$

implies that $\{x \in [0,1] \mid x \text{ is contained in infinitely many } E_n\} = \limsup_{n\to\infty} E_n \subset [0,1]\setminus N$. So we have

$$\limsup_{n\to\infty} m(E_n) \le \lim_{n\to\infty} m\left(\bigcup_{m=n}^{\infty} E_m\right) = m\left(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\right).$$

(The equality holds because $E_n \subset [0,1]$. See Corollary 2.8.) □

§ 2.3

31 (Lemma 2.10) Let $F \overset{\text{def}}{=} G^c$. Since $E_k \subset E_{k+1}$, $\lim_{k\to\infty} m^*(E_k)$ exists. When $\lim_{k\to} m^*(E_k) = \infty$, the statement holds obviously. We only need to consider $\lim_{k\to\infty} m^*(E_k) < \infty$.

STEP 1. (proof of $\bigcup_{k=1}^{\infty} E_k = E$) First we show that $\bigcup_{k=1}^{\infty} E_k = E$. Since $E_k \subset E$ for all $k = 1, 2 \cdots$, $\bigcup_{k=1}^{\infty} E_k \subset E$.

Next, for any $x \in E$, $\mathrm{dist}(x, G^c) > 0$. To verify this, let us recall Theorem 1.24. Since $F = G^c$ is a non-empty closed set, $\forall x \in \mathbb{R}^d$, there exists $y \in G^c = F$, $\mathrm{dist}(x, G^c) = |x - y|$. If $\mathrm{dist}(x, G^c) = 0$, then $x = y \in G^c$. However $x \in E \subset G$ so this contradicts to the assumption. So we conclude that $\mathrm{dist}(x, G^c) > 0$ for all $x \in E$. For each $x \in E$, by taking sufficiently large k, we have $\mathrm{dist}(x, G^c) \geqq \frac{1}{k}$. So $x \in \bigcup_{k=1}^{\infty} E_k$ for all $x \in E$. This implies that $E \subset \bigcup_{k=1}^{\infty} E_k$.

STEP 2. (proof of $\lim_{k\to\infty} m^*(E_k) = m^*(E)$) E_k is monotone increasing. So $\lim_{k\to\infty} m^*(E_k)$ exists. Obviously $\lim_{k\to\infty} m^*(E_k) \leqq m^*(E)$ holds. So our goal is to show that $m^*(E) \leqq \lim_{k\to\infty} m^*(E_k)$.

Let $A_k \overset{\text{def}}{=} E_k \setminus E_{k-1}$, $E_0 \overset{\text{def}}{=} \emptyset$. $\mathrm{dist}(A_{2k}, A_{2\ell}) > 0$ if $k < \ell$ holds. We will prove this later, but let us accept this fact for now. Since $\bigcup_{j=1}^{k} A_{2j} \subset E_{2k}$, we have

$$m^*\left(\bigcup_{j=1}^{k} A_{2j}\right) \leqq m^*(E_{2k}).$$

The left hand side is

$$m^*\left(\bigcup_{j=1}^{k} A_{2j}\right) = \sum_{j=1}^{k} m^*(A_{2j}),$$

because $\mathrm{dist}(A_{2k}, A_{2l}) > 0$ if $k < l$ and Theorem 2.4. Therefore,

$$\sum_{j=1}^{k} m^*(A_{2j}) \leqq m^*(E_{2k})$$

Similarly we also have

$$\sum_{j=1}^{k} m^*\left(A_{2j-1}\right) \leq m^*\left(E_{2k-1}\right).$$

By our assumption, $\sup_{k\geqq 1} m^*(E_k) = \lim_{k\to\infty} m^*(E_k) < \infty$, therefore we have

$$\sum_{k=1}^{\infty} m^*(A_{2k}), \sum_{k=1}^{\infty} m^*(A_{2k-1}) < \infty.$$

Since

$$E = E_{2k} \cup \bigcup_{j=k+1}^{\infty} A_{2j} \cup \bigcup_{j=k+1}^{\infty} A_{2j-1},$$

and by sub-additivity of an outer measure, we have

$$m^*(E) \leqq m^*(E_{2k}) + \sum_{j=k+1}^{\infty} m^*(A_{2j}) + \sum_{j=k+1}^{\infty} m^*(A_{2j-1}).$$

By taking $k \to \infty$, $\sum_{j=k+1}^{\infty} m^*(A_{2j}) + \sum_{j=k+1}^{\infty} m^*(A_{2j-1}) \to 0$. So we conclude that

$$m^*(E) \leqq \lim_{k\to\infty} m^*(E_{2k}).$$

Since E_k is monotone increasing, so the right hand side $= \lim_{k\to\infty} m^*(E_k)$.

STEP 3. (proof of $\mathrm{dist}(A_{2k}, A_{2\ell}) > 0, k < \ell$) Let $x_1 \in A_{2k}, x_2 \in A_{2\ell}$. Sinde $F = G^c$ is a non-empty closed set, there exists $y_2 \in G^c = F$ s.t

$$|x_2 - y_2| = \mathrm{dist}(x_2, G^c).$$

By triangular inequality, we have

$$\begin{aligned} \mathrm{dist}(x_1, x_2) &\geqq \mathrm{dist}(x_1, y_2) - \mathrm{dist}(x_2, y_2) \\ &= \mathrm{dist}(x_1, y_2) - \mathrm{dist}(x_2, G^c) \end{aligned}$$

Further more, since $\mathrm{dist}(x_1, y_2) \geqq \mathrm{dist}(x_1, G^c) \stackrel{\text{def}}{=} \inf_{y \in G^c} |x_1 - y|$, we have

$$\mathrm{dist}(x_1, x_2) \geqq \mathrm{dist}(x_1, G^c) - \mathrm{dist}(x_2, G^c).$$

Since $x_1 \in A_{2k}, x_2 \in A_{2\ell}$, $\mathrm{dist}(x_1, G^c) \geqq \frac{1}{2k}$ and $\mathrm{dist}(x_1, G^c) < \frac{1}{2\ell-1}$, so we have

$$\begin{aligned} |x_1 - x_2| &= \mathrm{dist}(x_1, x_2) \\ &\geqq \mathrm{dist}(x_1, G^c) - \mathrm{dist}(x_2, G^c) \\ &\geqq \frac{1}{2k} - \frac{1}{2\ell - 1}. \end{aligned}$$

This implies that

$$\inf_{x_1 \in A_{2k}, x_2 \in A_{2\ell}} |x_1, x_2| \geqq \frac{1}{2k} - \frac{1}{2\ell - 1} > 0.$$

□

32 (Theorem 2.11) Let B an arbitrary subset of$\mathbb{R}^d$ and let F be a non-empty closed set. We use Lemma 2.10 ($G = F^c, E = B \setminus F \subset G$). Let

$$E_k \stackrel{\text{def}}{=} \left\{ x \in B \setminus F \mid \mathrm{dist}(x, F) \geqq \frac{1}{k} \right\}.$$

Then $\lim_{k \to \infty} m^*(E_k) = m^*(B \setminus F)$. Since

$$\begin{aligned} m^*(B) &= m^*(B \cap F \cup B \setminus F) \\ &\stackrel{*1}{\geqq} m^*(B \cap F \cup E_k) \\ &\stackrel{*2}{=} m^*(B \cap F) + m^*(E_k), \end{aligned}$$

- (∗1) $B \setminus F \supset E_k$.
- (∗2) This hold because $\mathrm{dist}(E_k, B \cap F) > 0$. First, $\mathrm{dist}(E_k, B \cap F) \geqq \mathrm{dist}(E_k, F)$. (It is easy to verify by the definition of $\mathrm{dist}(\cdot, \cdot)$.) Let $x \in E_k, y \in F$. be arbitrary points in E_k and F. Then $|x - y| \geqq \mathrm{dist}(x, F) \geqq \frac{1}{k}$. Therefore $\mathrm{dist}(E_k, F) \geqq \frac{1}{k}$.

Finally,by taking $k \nearrow \infty$, we have the desired result. □

33 (Theorem 2.12) Let $\mathscr{O}^d$ be a collection of open sets on $\mathbb{R}^d$ and let $\mathscr{B}$ be a family of Borel measurable sets. $\forall G \in \mathscr{O}^d, F \overset{\text{def}}{=} G^c \in \mathscr{M} \Rightarrow G \in \mathscr{M}$ so $\mathscr{O}^d \subset \mathscr{M}$. Since $\mathscr{B} \overset{\text{def}}{=} \sigma[\mathscr{O}^d]$ is the smallest $\sigma-$algera which contains $\mathscr{O}^d$, $\sigma[\mathscr{O}^d] \subset \mathscr{M}$. □

34 (Theorem 2.13)

(1)

case 1. $(m(E) < \infty)$ By the definition of Lebesgue (outer) measure, we have $\{I_n\}_{n=1}^{\infty}, E \subset \bigcup_{n=1}^{\infty} I_n$, s.t

$$m(E) \leqq \sum_{n=1}^{\infty} |I_n| < m(E) < \epsilon$$

Let $G \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} I_n$. We show that G is the desired open set. By sub additivity $m(G) \leqq \sum_{n=1}^{\infty} |I_n| < m(E) + \epsilon$. Since $m(G) < \infty$ and $E \subset G$, $m(G \setminus E) = m(G) - (E) < \epsilon$.

case 2. $(m(E) = \infty)$ Let $E_k \overset{\text{def}}{=} E \cap B(0, k)$ $(E = \bigcup_{k=1}^{\infty} E_k)$. Then $m(E_k) < \infty$. From the previous result, for each E_k we have an open set $G_k \supset E_k$ s.t , $m(G_k \setminus E_k) < \frac{\epsilon}{2^k}$. Let $G \overset{\text{def}}{=} \bigcup_{k=1}^{\infty} G_k$. G is the desired open set. $m(G \setminus E) \leqq \sum_{k=1}^{\infty} m(G_k \setminus E_k) \leqq \epsilon$.

(2) We have $G \supset E^c$. s.t $m(G \setminus E^c) < \epsilon$ from the previous result. Let $F \overset{\text{def}}{=} G^c$. Then $m(E \setminus F) = m(G \setminus E^c) < \epsilon$.

□

35 (Converse of Theorem 2.13) We can find a sequence of open sets $\{G_n\}_{n\geq 1}^{\infty}$ s.t $m^*(G_n \setminus E) < \frac{1}{n}$. Let $H \overset{\text{def}}{=} \bigcap_{n=1}^{\infty} G_n \in \mathscr{M}$. Then $m^*(H \setminus E) \leqq m^*(G_n \setminus E) < \frac{1}{n}$ for all $n = 1, 2 \cdots$ so $m(H \setminus E) = 0$. Finally $E = H \setminus (H \setminus E) \in \mathscr{M}$ because $H, H \setminus E \in \mathscr{M}$. Now the proof is complete. □

36 (Theorem 2.14)

(1) By Theorem 2.13, we have G_n : an open set s.t $m(G_n \setminus E) < \frac{1}{n}$ and $E \subset G_n$. Let $H \overset{\text{def}}{=} \bigcap_{n=1}^{\infty} G_n$. (This is a G_δ set.) Then $E \subset H$ and $m(H \setminus E) \leqq m(G_n \setminus E) < \frac{1}{n}$ for all $n = 1, 2 \cdots$. So $m(H \setminus E) = 0$. Let $Z \overset{\text{def}}{=} H \setminus E$. (This is a measure zero set.) Then $E = H \setminus Z$.

(2) By Theorem 2.13, we have F_n : a closed set s.t $m(E \setminus F_n) < \frac{1}{n}$. Let $K \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} F_n$. (This is a F_σ set.) Then $m(E \setminus K) \leqq m(E \setminus F_n) < \frac{1}{n}$ for all $n = 1, 2 \cdots$. So $m(E \setminus K) = 0$. Finally let $Z \overset{\text{def}}{=} E \setminus K$. (This is a measure zero set). Then $E = K \cup Z$

□

37 (Theorem 2.15) We may suppose that $m^*(E) < \infty$ because if $m^*(E) = \infty$, $\mathbb{R}^d$ is the desied set. By the definition of Lebesgue outer measure, for each $n = 1, 2 \cdots$, we have $\{I_{n,k}\}_{k \geqq 1}$ s.t

$$m^*(E) \leqq \sum_{k=1}^{n} |I_{n,k}| < m^*(E) + \frac{1}{n}.$$

Let $G_n \stackrel{\text{def}}{=} \bigcup_{k=1}^{\infty} I_{n,k}$. (This is an open set. $G_n \supset E$) Then we have

$$m(G_n) \leqq m^*(E) + \frac{1}{n}.$$

Finally let $H \stackrel{\text{def}}{=} \bigcap_{n=1}^{\infty} G_n$. (This is a G_δ set. $H \supset E$) Then we have

$$m^*(E) \leqq m(H) \leqq m(G_n) \leqq m^*(E) + \frac{1}{n}, \forall n = 1, 2 \cdots .$$

So $m^*(E) = m(H)$. □

38 (Corollary 2.16 and Corollary 2.17)

(1) For each $k = 1, 2 \cdots$, we take a G_δ-set H_k s.t $E_k \subset H_k$ and $m^*(E_k) = m(H_k)$.

$$\begin{aligned}
m^*(\liminf_{k\to\infty} E_k) &\stackrel{*1}{\leqq} m\left(\liminf_{k\to\infty} H_k\right) \\
&\stackrel{*2}{=} m\left(\bigcup_{k=1}^{\infty}\bigcap_{m=k}^{\infty} H_m\right) \\
&\stackrel{*3}{=} \lim_{k\to\infty} m\left(\bigcap_{m=k}^{\infty} H_m\right) \\
&\stackrel{*4}{\leqq} \liminf_{k\to\infty} m(H_k) \\
&\stackrel{*5}{=} \liminf_{k\to\infty} m^*(E_k).
\end{aligned}$$

- $(*1)$ $E_k \subset H_k$
- $(*2)$ By definition.
- $(*3)$ $\bigcap_{m=k}^{\infty} H_m$ is an inreasing sequence of sets with respect to k. So we may swap lim and m.
- $(*4)$ $\bigcap_{m=k}^{\infty} H_m \subset H_k$. $m(H_k)$ does not necessarily have a limit. So we consider $\liminf$.
- $(*5)$ $m^*(E_k) = m(H_k)$.

So we have the desired result. Notice. Some people may think that we can use this lemma to prove Lemma 2.10. But we can not do so. In this proof, we used measurability of H_k which was derived from the fact that a closed set is Lebesgue measurable. However measurability of closed sets was proved using Lemma 2.10.

(2) If E_k is an increasing sequence, each $\liminf$ in the formula above becomes lim So we have $m^*(\lim_{k\to\infty} E_k) \leqq \liminf_{k\to\infty} m^*(E_k)$. $\geqq$ is obvious.

□

39 (Theorem 2.18 (a)) We have already shown that $m^*(E) = m^*(E_{+x_0})$ where $E_{+x_0} = \{x + x_0 | x \in E\}$. In this theorem we prove measurability is also preserved by translation. Suppose $E \in \mathscr{M}$. Then $E = H \setminus Z$ where H is a G_δ set and Z is a measure zero set. Obviously $E_{+x_0} = H_{+x_0} \setminus Z_{+x_0}$. Z_{+x_0} is also a measure zero set since translation does not change outer measure. $H_{+x_0} = \bigcap_{k=1}^{\infty} G_{k+x_0}$. Obviously G_{k+x_0} is also an open set. So E_{+x_0} is also measurable. □

40 (Theorem 2.18 (b)) Let $E \subset \mathbb{R}$. Let us recall that $\forall \lambda \in \mathbb{R}$, $m^*(\lambda E) = |\lambda| m^*(E)$. (Theorem 2.5 (b), $d = 1$.) Let B be an arbitrary set on $\mathbb{R}$.

STEP 1.

$$\begin{aligned}
& m^*(B \cap \lambda E) + m^*(B \cap (\lambda E)^c) \\
\overset{*1}{=} \; & m^*(B \cap \lambda E) + m^*(B \cap \lambda(E^c)) \\
\overset{*2}{=} \; & m^*(\lambda(\lambda^{-1} B \cap E)) + m^*(\lambda(\lambda^{-1} B \cap E^c)) \\
= \; & |\lambda| m^*(\lambda^{-1} B \cap E) + |\lambda| m^*(\lambda^{-1} B \cap E^c).
\end{aligned}$$

- $(*1)$ $(\lambda E)^c = \lambda(E^c)$holds. We explain this in the next step.
- $(*2)$ $\lambda(A \cap B) = \lambda A \cap \lambda B$ holds. We also explain this in the next step.

Further more, since E is mesurable, for all $\tilde{B} \subset \mathbb{R}$, we have

$$m^*(\tilde{B} \cap E) + m^*(\tilde{B} \cap E^c) \leqq m^*(\tilde{B}).$$

Let $\tilde{B} = \lambda^{-1} B$. Then we have

$$\begin{aligned}
& |\lambda| m^*(\lambda^{-1} B \cap E) + |\lambda| m^*(\lambda^{-1} B \cap E^c) \\
\leqq \; & |\lambda| m^*(\lambda^{-1} B) = |\lambda| \cdot \frac{1}{|\lambda|} m^*(B) = m^*(B).
\end{aligned}$$

Now the proof is complete.

STEP 2. First, we verify that $(\lambda E)^c = \lambda E^c$. Let $f(x) \overset{\text{def}}{=} \frac{x}{\lambda}$. Then $\lambda E = f^{-1}(E)$. Since $f^{-1}(E)^c = f^{-1}(E^c)$, we have $(\lambda E)^c = \lambda E^c$.

Next, let $f(x) \overset{\text{def}}{=} \lambda x$. $f(A \cap B) = f(A) \cap f(B)$ holds. So $\lambda A \cap \lambda B = \lambda(A \cap B)$

□

41 (Exercise 1) Cosider a G_δ set $H_1 \supset E$ s.t $m(H_1) = m^*(E) < \infty$. Let $\{F_k\}; F_k \subset E$ be a bounded closed set with $m(F_k) \nearrow m^*(E)$. Let $H_2 \overset{\text{def}}{=} \bigcup_{k=1}^{\infty} F_k$. $m^*(E \setminus H_2) \leqq m(H_1 \setminus H_2) = m(H_1) - m(H_2) \leqq m^*(E) - m(F_k)$ for all $k = 1, 2 \cdots$. So $m^*(E \setminus H_2) = 0 \Rightarrow E \setminus H_2 \in \mathscr{M}$. Finally $E = (E \setminus H_2) \cup H_2 \in \mathscr{M}$. □

42 (Exercise 2)

(1) We prove the contraposition. Suppose that $\overline{E} \neq [0,1]$. Let us pick $x_0 \in [0,1] \setminus \overline{E}$. Since $x_0 \in [0,1] \setminus E'$, there exists $\delta_0 > 0$ s.t $B(x_0, \delta_0) \cap E = \emptyset$. Therefore, $B(x_0, \delta_0) \cap [0,1] \subset [0,1] \setminus E$. And we have

$$0 < m(B(x_0, \delta_0) \cap [0,1]) \leqq m([0,1] \setminus E) = m([0,1]) - m(E).$$

So $m(E) < 1$.

(2) We prove the contraposition. Suppose that $\mathring{E} \neq \emptyset$. This implies that $\exists x_0 \in E$ and $\exists \delta_0 > 0$ s.t $B(x_0, \delta_0) \subset E$. Then $0 < m(B(x_0, \delta_0)) \leqq m(E)$.

□

43 (Exercise 3) We prove the contraposition. In other words, our goal is to prove that if $\exists x_0 \in (a,b)$ s.t $f(x_0) > g(x_0)$ then $\exists t_0 \in \mathbb{R}$ s.t $m(\{x \in [a,b] \mid f(x) > t_0\}) > m(\{x \in [a,b] \mid g(x) > t_0\})$.

Let $t_0 = f(x_0)$, then t_0 is the desired $t_0 \in \mathbb{R}$. $m(\{x \in [a,b] \mid f(x) > t_0\}) = m([a, x_0)) = x_0 - a$ because $f(x)$ is continuous and strictly decreasing. Since $g(x)$ is also continuous and strictly decreasing, there exists $\delta_0 > 0$ s.t

$$\forall x \in (x_0 - \delta_0, b],\ g(x) < t_0 = f(x_0).$$

So

$$\begin{aligned} m(\{x \in [a,b] \mid g(x) > t_0\} &\leqq m(\{x \in [a,b] \mid g(x) \geqq t_0\}) \\ &\leqq m([a, x_0 - \delta]) = x_0 - a - \delta. \\ &< x_0 - a \\ &= m(\{x \in [a,b] \mid f(x) > t_0\}) \end{aligned}$$

□

44 (Exercise 4) We use Theorem 2.11 and Theorem 2.13. (Recall that we have not shown that a closed set is Lebesgue measurable.)

STEP 1. First we explain that we may suppose that E is bounded without loss of generality. Let $E_n \overset{\text{def}}{=} E \cap [-n, n]$. Then $m(E_n) \nearrow m(E)$. If $m(E) > \alpha$, we can find n s.t $m(E_n) > \alpha$. So we just need to find $F \subset E_n$ s.t $m(F) = \alpha$. We explain how to find such F in the next step.

STEP 2. Next we suppose that $E \subset [-M, M]$ is bounded. By Theorem 2.13, there exists a closed set $K \subset E$ s.t $m(E \setminus K) < \epsilon \overset{\text{def}}{=} m(E) - \alpha$. Since $m(E) < \infty$ ($\because$ bounded), $m(E \setminus K) = m(E) - m(K) < m(E) - \alpha$. So $m(K) > \alpha$. Now let $f(x) \overset{\text{def}}{=} m(K \cap [-M, x])$. Then $f(x)$ is continuous because $f(x+h) - f(x) \leqq m((x, x+h]) = h$. $f(-M) = 0$ and $f(M) = m(K) > \alpha$. By intermediate value theorem, there exists $c \in [-M, M]$ s.t $f(c) = \alpha$. So $F \overset{\text{def}}{=} K \cap [-M, c]$ is the desired closed set.

□

45 (Exercise 5) This does not necessarily hold. Let

$$G \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} \left(r_n - \frac{1}{2^{n+1}}, r_n + \frac{1}{2^{n+1}} \right),$$

where $\{r_n\} \stackrel{\text{def}}{=} \mathbb{Q}$. Since G contains all rational numbers on $\mathbb{R}$ (hence it is dense), so $\overline{G} = \mathbb{R}$, however

$$m(G) \leqq \sum_{n=1}^{\infty} m\left(\left(r_n - \frac{1}{2^{n+1}}, r_n + \frac{1}{2^{n+1}} \right) \right) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

□

46 (Exercise 6) Let us consider G_δ sets H_1, H_2 s.t $E_1 \subset H_1, E_2 \subset H_2$ and

$$m^*(E_1) = m(H_1),\ m^*(E_2) = m(H_2).$$

We have

$$\begin{aligned} m(H_1) + m(H_2) \stackrel{*1}{\geqq} m(H_1 \cup H_2) &\stackrel{*2}{\geqq} m(E_1 \cup E_2) \\ &\stackrel{*3}{=} m^*(E_1) + m^*(E_2) \\ &\stackrel{*4}{=} m(H_1) + m(H_2). \end{aligned}$$

- ($*1$) sub-additivity
- ($*2$) monotonicity of measure
- ($*3$) by assumption
- ($*4$) $m(H_1) = m^*(E_1)$, $m(H_2) = m^*(E_2)$

From this fact, we find out that

$$m(H_1 \cap H_2) = 0,$$

hence

$$m^*(E_1 \cap E_2) = 0.$$

So $E_1 \cap E_2$ is also a measure zero set hence $E_1 \cap E_2 \in \mathscr{M}$. Moreover

$$m(H_1 \cup H_2 \setminus (E_1 \cup E_2)) \stackrel{*5}{=} m(H_1 \cup H_2) - m(E_1 \cup E_2) = 0,$$

so $H_1 \cup H_2 \setminus (E_1 \cup E_2)$ is also a measure zero set.

- ($*5$) Both $H_1 \cup H_2$, $E_1 \cup E_2$ are measurable and $H_1 \cup H_2 \supset E_1 \cup E_2$ and $m(E_1 \cup E_2) < \infty$.

Therefore we find out that both $H_1 \setminus E_1$, $H_2 \setminus E_2$ are measure zero sets. (You may draw a Ben figure.) Finally $E_1 = H_1 \setminus (H_1 \setminus E_1)$ and $E_2 = H_2 \setminus (H_2 \setminus E_2)$. So we have the desired conclusion. □

47 (Exercise 7)

STEP 1. Let $\{r_n\} \stackrel{\text{def}}{=} [0,1] \cap \mathbb{Q}$, let $I_{n,k} \stackrel{\text{def}}{=} B(r_n, \frac{1}{2^{n+k}})$ and let $G_k \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} I_{n,k}$. G_k contains all rational numbers in $[0,1]$ so G_k is dense in $[0,1]$. $m\left(\bigcap_{k=1}^{\infty} G_k\right) \leqq m(G_k) \leqq \sum_{n=1}^{\infty} \frac{1}{2^{n+k-1}} = \frac{1}{2^{k-1}}$. So $m\left(\bigcap_{k=1}^{\infty} G_k\right) = 0$. We prove that $E \stackrel{\text{def}}{=} \bigcap_{k=1}^{\infty} G_k$ is a set of the second category. (not a meagre set).

STEP 2. Let $E^c \stackrel{\text{def}}{=} [0,1] \backslash E = \bigcup_{k=1}^{\infty} [0,1] \backslash G_k$. Since $[0,1] \backslash G_k$ is closed, $\overline{[0,1] \setminus G_k} = [0,1] \setminus G_k$. Therefore $\overset{\circ}{\overline{[0,1] \setminus G_k}} = \overset{\circ}{[0,1] \setminus G_k}$. It is enough to show that $[0,1] \setminus G_k$ has no interior point.

Since G_k is dense in $[0,1]$ (G_k contains all rational numbers in $[0,1]$), $[0,1] \setminus G_k$ has no interior point . So $\overset{\circ}{\overline{[0,1] \setminus G_k}} = \overset{\circ}{[0,1] \setminus G_k} = \emptyset$, hence $[0,1] \setminus G_k$ is a nowhere dense set. Therefore $E^c = \bigcup_{k=1}^{\infty} [0,1] \setminus G_k$ is a meagre set (a set of the first category). So E is a set of the second category. $(*)$

STEP 3. Finally, we explain $(*)$. We show that if $A \subset \mathbb{R}^d$ is a meagre set. Then $B \stackrel{\text{def}}{=} A^c$ is not a meagre set (a set of the second category). Suppose both A, B are meagre sets and $A = \bigcup_{n=1}^{\infty} F_n^{(1)}, B = \bigcup_{n=1}^{\infty} F_n^{(2)}$, where $\{F_n^{(1)}\} \cup \{F_n^{(2)}\}$ are collections of nowhere dense sets.

$$\begin{aligned}
\mathbb{R}^d = A \cup B &= \bigcup_{n=1}^{\infty} F_n^{(1)} \cup \bigcup_{n=1}^{\infty} F_n^{(2)} \\
&\subset \bigcup_{n=1}^{\infty} \overline{F}_n^{(1)} \cup \bigcup_{n=1}^{\infty} \overline{F}_n^{(2)} \\
&= \mathbb{R}^d
\end{aligned}$$

So it follow that $\mathbb{R}^d$ does not have an interior point by Baire's theorem (Theorem 1.23). (contradiction!)

□

48 (Exercise 8)

STEP 1. Let

$$A \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R} \mid \# \left\{ (p,q) \in \mathbb{Z} \times \mathbb{N} \ : \ \left| x - \frac{p}{q} \right| \leqq \frac{1}{q^3} \right\} = +\infty \right\}.$$

We show that $m(A) = 0$. Let $B_n \stackrel{\text{def}}{=} [n-1, n] \cap A, n \in \mathbb{Z}$. It is enough for us to prove that $m(B_1) = 0$ because for any $m \in \mathbb{Z}$, $x \in A \Rightarrow x + m \in A$ and this implies $m(B_0) = m(B_1) = m(B_{-1}) = \cdots$. ($|x - \frac{p}{q}| = |x + m - \frac{p+mq}{q}| = |x + m - \frac{p'}{q}|$.)

STEP 2. Let $B = B_1$. We show that $m(B) = 0$. Let

$$I_{p,q} \stackrel{\text{def}}{=} \left(\frac{p}{q} - \frac{1}{q^3}, \frac{p}{q} + \frac{1}{q^3} \right)$$

Suppose $x \in B \subset [0,1]$. There are infinitely many $(p,q) \in \mathbb{Z} \times \mathbb{N}$ s.t $|x - \frac{p}{q}| \leqq \frac{1}{q^3} (\Leftrightarrow x \in I_{p,q})$. So

$$qx - \frac{1}{q^2} \leqq p \leqq qx + \frac{1}{q^2}.$$

Moreover, since $x \in [0,1], q \geqq 1$,

$$-1 \leqq p \leqq q+1.$$

From this inequality, we can find out that there are only finite number of p s.t $x \in I_{p,q}$ for each fixed $q = 1, 2 \cdots$. So if we let $E_q \overset{\text{def}}{=} \bigcup_{p=-1}^{q+1} I_{p,q}$, there should be infinitely many q s.t $x \in E_q$. Therefore $\forall x \in B$, we have

$$x \in \{x \in \mathbb{R} \mid x \text{ is contained in infinitely many } E_q\} = \limsup_{q \to \infty} E_q$$

In other words,

$$B \subset \limsup_{q \to \infty} E_q$$

Now $\sum_{q=1}^{\infty} m(E_q) < \infty$, $(\because m(E_q) \leqq \frac{q+2}{q^3})$,

$$\begin{aligned} m\left(\limsup_{q \to \infty} E_q\right) &= \lim_{q \to \infty} m\left(\bigcup_{m=q}^{\infty} E_m\right) \\ &\leqq \lim_{q \to \infty} \sum_{m=q}^{\infty} m(E_m) = 0. \end{aligned}$$

(You may also use Borel-Cantelli's lemma to explain this part.)

□

§ 2.4

49 (Theorem 2.19)

STEP 1. $(m(E) = +\infty)$ In this theorem, we may suppose that $m(E) < \infty$. Let $E_k \overset{\text{def}}{=} B(0,k) \cap E$. Then $E_k \nearrow E$, so we may find k s.t $0 < m(E_k) < \infty$. So let us find the desired interval with respect to E_k. Then, $\lambda |I| < m(I \cap E_k) \leqq m(I \cap E)$.

STEP 2. $(m(E) < \infty)$ Let

$$\epsilon \in \left(0, \left(\frac{1}{\lambda} - 1\right) \cdot m(E)\right).$$

We may find a collection of open intervals $\{I_k\}_{k \geqq 1}$ s.t

$$E \subset \bigcup_{k=1}^{\infty} I_k, \ \sum_{k=1}^{\infty} |I_k| < m(E) + \epsilon.$$

Next if we suppose that

$$m(E \cap I_k) \leqq \lambda |I_k|, \ \forall k = 1, 2 \cdots, \ (*)$$

then since $E = \bigcup_{k=1}^{\infty} E \cap I_k$, we have

$$\begin{aligned} m(E) &\overset{*1}{\leqq} \sum_{k=1}^{\infty} m(E \cap I_k) \\ &\overset{*2}{\leqq} \sum_{k=1}^{\infty} \lambda |I_k| \\ &\overset{*3}{<} \lambda(m(E) + \epsilon) = \lambda m(E) + \lambda \epsilon \\ &\overset{*4}{<} \lambda m(E) + \lambda \cdot \left(\frac{1}{\lambda} - 1\right) \cdot m(E) \\ &< m(E). \end{aligned}$$

- $(*1)$ measure has sub-additivity
- $(*2)$ we suppose that $m(E \cap I_k) \leqq \lambda |I_k|$
- $(*3)$ we picked $\{I_k\}_{k=1}^{\infty}$ s.t $\sum_{k=1}^{\infty} |I_k| < m(E) + \epsilon$
- $(*4)$ we chose $\epsilon < (\frac{1}{\lambda} - 1) \cdot m(E)$

A contradiction occured because the assumption $(*)$ is incorrect. So there exists at least one I_{k_0} s.t

$$m(E \cap I_{k_0}) > \lambda |I_{k_0}|.$$

□

50 (Theorem 2.20 Steinhaus Theorem)

STEP 1. Let

$$\lambda \in \left(1 - \frac{1}{2^{n+1}}, 1\right).$$

By Theorem 2.19, we find an open interval $I \overset{\text{def}}{=} \prod_{i=1}^{d} (a_i, b_i)$ s.t

$$\lambda |I| < m(E \cap I). \ (*a)$$

Now let δ be the shortest edge length of I.

$$\delta \overset{\text{def}}{=} \min_{i=1 \cdots d} \{b_i - a_i\}$$

STEP 2. Let

$$J \overset{\text{def}}{=} \prod_{i=1}^{d} \left(-\frac{\delta}{2}, \frac{\delta}{2}\right).$$

We prove that $J \subset E - E$. However, it is enough for us to prove the following statement.

$$\forall x_0 \in J, \ (E \cap I) \cap (E \cap I)_{+x_0} \neq \emptyset. \ (*b)$$

We state the reason why it is enough for us to prove $(*b)$. If there exists $y \in (E \cap I) \cap (E \cap I)_{+x_0}$, then $y \in (E \cap I)$ and $y = z + x_0$ for some $z \in (E \cap I)$. This means that $\exists y, z \in (E \cap I)$ s.t $y - z = x_0$. In other words, $x_0 \in E \cap I - E \cap I \overset{\text{def}}{=} \{y - z \mid y, z \in E \cap I\}$ for all $x_0 \in J$. So $J \subset E \cap I - E \cap I$. Moreover, obviously, $E \cap I - E \cap I \subset E - E$. So we can conclude that $J \subset E - E$.

STEP 3. (Proof of $(*b)$) Let $x_0 = (x_{0,1}, \cdots x_{0,d})$. Since $|x_{0,i}| < \frac{\delta}{2}$ and δ is the shortest edge length of I, the each edge length of $I \cap I_{+x_0}$ is larger than $\frac{1}{2}(b_i - a_i)$. So

$$m(I \cap I_{+x_0}) > \frac{1}{2^d} \cdot |I| \,. \; (*c)$$

And

$$\begin{aligned} m(I \cup I_{+x_0}) &\overset{*1}{=} m(I) + m(I_{+x_0}) - m(I \cap I_{x+0}) \\ &\overset{*2}{=} |I| + |I_{+x_0}| - m\left(I \cap I_{+x_0}\right) \\ &\overset{*3}{=} 2\,|I| - m\left(I \cap I_{+x_0}\right) \\ &\overset{*4}{<} 2\,|I| - \frac{1}{2^{d+1}} \cdot |I| \\ &= 2\left(1 - \frac{1}{2^{d+1}}\right)|I| \\ &\overset{*5}{<} 2\lambda\,|I| \,. \end{aligned}$$

- $(*1)$ $m(A \cup B) = m(A) + m(B) - m(A \cap B)$.
- $(*2)$ $m(I) = |I|$ when I is an open rectangle.
- $(*3)$ obviously $|I| = |I_{+x_0}|$ holds from the definition of $|I|$.
- $(*4)$ by $(*c)$
- $(*5)$ we assume $1 - \frac{1}{2^{d+1}} < \lambda$.

Now suppose $E \cap I$ and $(E \cap I)_{+x_0}$ are disjoint. Then,

$$\begin{aligned} m\left((E \cap I) \cup (E \cap I)_{+x_0}\right) &= m(E \cap I) + m((E \cap I)_{+x_0}) \\ &= 2m(E \cap I) \\ &\overset{*6}{\leqq} m\left(I \cup I_{+x_0}\right) \\ &< 2\lambda\,|I| \,. \end{aligned}$$

- $(*6)$ $E \cap I \bigcup (E \cap I)_{+x_0} \subset I \cup I_{+x_0}$

So we have $m(E \cap I) < \lambda\,|I|$. However this contradicts to the assumption $(*a)$. So $E \cap I$ and $(E \cap I)_{+x_0}$ are not disjoint for any $x_0 \in J$. Now the proof of $(*b)$ is complete.

□

51 (Exercise 1) By Theorem 2.20,

$$(-a,a) \subset E - E, \text{ for some } a > 0.$$

This means that $\forall x \in (-a,a)$, $\exists y, z \in E$ s.t $x = y - z$. Since $y = x + z$, $y \in E_{+x}$. So

$$y \in E \cap E_{+x} \neq \emptyset.$$

□

52 (Exercise 2) By assumption, for all $x \in (-\delta,\delta)$, $x \in E_{-a}$ or $x \in -E_{-a}$ where $-E_{-a} \overset{\text{def}}{=} \{-x \mid x \in E_{-a}\}$. Therefore we have

$$(-\delta,\delta) \subset E_{-a} \cup -E_{-a}.$$

By monotonicity and sub-additivity of Lebesgue measure, we have

$$\begin{aligned} m((-\delta,\delta)) &\overset{*1}{\leqq} m^*\left(E_{-a} \cup -E_{-a}\right) \\ &\overset{*2}{\leqq} m^*(E_{-a}) + m^*(-E_{-a}) \\ &\overset{*3}{=} m^*(E_{-a}) + m^*(E_{-a}) \\ &= 2m^*(E_{-a}) \\ &\overset{*4}{=} 2m(E_{-a}) \\ &\overset{*5}{=} 2m(E) \end{aligned}$$

- $(*1)$ $A \subset B$ then $m^*(A) \leqq m^*(B)$
- $(*2)$ $m^*(A \cup B) \leqq m^*(A) + m^*(B)$
- $(*3)$ $m^*(\lambda E) = |\lambda|^d m^*(E)$. (Theorem 2.5)
- $(*4)$ By Theorem 2.18, $E \in \mathscr{M}$ then $E_{-a} \in \mathscr{M}$. So we can change m^* to m.
- $(*5)$ By Theorem 2.18 or Theorem 2.5. Translation does not change the value of Lebesgue outer measure.

From the inequality above, it follows that $2\delta \leqq 2m(E)$. Now the proof is complete. □

53 (Exercise 3) f is bounded in E. We suppose that $|f(x)| \leqq M$ for all $x \in E$.

STEP 1. $(\forall r \in \mathbb{Q}, f(r) = rf(1))$ First $f(0) = 2f(0)$, so $f(0) = 0$. Next $y = -x$ and we have $f(x) = -f(x)$. Then $f(n) = nf(1)$ for $n \in \mathbb{Z}$.

Let $r \in \mathbb{Q}$. Then $r = \frac{n}{m}, n \in \mathbb{Z}, m \in \mathbb{N}$. $f(\frac{n}{m} + \frac{n}{m} \cdots \frac{n}{m}) = mf(\frac{n}{m}) = f(n) = nf(1)$. So $f(\frac{n}{m}) = \frac{n}{m}f(1)$.

STEP 2. By Theorem 2.20, there exists an interval $I = [-c,c] \subset E - E$. Let $x \in I$. Then $\exists x_1, x_2 \in E$ s.t $x = x_1 - x_2$. By assumption, $f(x) = f(x_1) - f(x_2)$ so $|f(x)| = |f(x_1) - f(x_2)| \leqq 2M < \infty$.

STEP 3. Let $x \in \mathbb{R}$ and let $N \in \mathbb{N}$. We can always find $r \in \mathbb{Q}$ s.t $|x - r| \leqq \frac{c}{N}$ because $\mathbb{Q}$ is dense in $\mathbb{R}$. We show that $|f(x) - xf(1)| = 0$ for all $x \in \mathbb{R}$.

$$\begin{aligned} |f(x) - xf(1)| &= |f(x-r) + f(r) - xf(1)| \\ &= |f(x-r) + rf(1) - xf(1)| \\ &\leqq |f(x-r)| + |r - x|\,|f(1)| \\ &\leqq |f(x-r)| + \frac{c}{N}\,|f(1)|\,. \end{aligned}$$

Moreover,

$$\begin{aligned} |f(x-r)| &= \left| f\left(\frac{1}{N} \cdot N(x-r)\right)\right| \\ &= \frac{1}{N}\,|f\left(N(x-r)\right)| \\ &\overset{*}{\leqq} \frac{2M}{N}. \end{aligned}$$

- $(*)$ $N(x-r) \in I = [-c, c]$ so $|f(N(x-r))| \leqq 2M$ by STEP 2.

Since $N \in \mathbb{N}$ is arbitrary, the right hand side $\searrow 0$ by taking $N \nearrow \infty$.

□

§ 2.5

54 (Example: non Lebesgue measurable set)

(1) First we construct a non Lebesgue measurable set on $\mathbb{R}^1$.

STEP 1. Let

$$\Gamma_x \overset{\text{def}}{=} \{x + r \mid r \in \mathbb{Q}\}\ (x \in \mathbb{R}),\ \mathbb{R} \setminus \mathbb{Q} \overset{\text{def}}{=} \{\Gamma_x \mid x \in \mathbb{R}\}\,.$$

By axiom choice, from each $\Gamma \in \mathbb{R} \setminus \mathbb{Q}$, we can pick an element $a \in \Gamma$, and define a new set $W \overset{\text{def}}{=} \{a\}$ by gathering $a \in \Gamma$ together. Note the following facts.

- (*1) If $x - y \in \mathbb{Q}$, then $\Gamma_x = \Gamma_y$. (In this case, Γ_x, Γ_y are equivalent.)
- (*2) If $a_1, a_2 \in W$ $(a_1 \neq a_2)$, then $a_1 - a_2 \notin \mathbb{Q}$. (This implies that $W - W$ does not contain any rational numbers except for 0.)
- (*3) $\mathbb{R} = \bigcup_{r \in \mathbb{Q}} W_{+r}$ where $W_{+r} \overset{\text{def}}{=} \{x + r \mid x \in W\}$.

(*1) is easy to verify. (*2) If $a_1 - a_2 \in \mathbb{Q}$, then there exists $\Gamma \in \mathbb{R} \setminus \mathbb{Q}$ s.t $a_1, a_2 \in \Gamma$. But we pick an element $a \in \Gamma$ only once from each $\Gamma \in \mathbb{R} \setminus \mathbb{Q}$, so both $a_1, a_2 \in \Gamma$ can not be contained in W. (*3) Let us pick an arbitrary real number $x \in \mathbb{R}$. There exists some $a \in W$ s.t $x \in \Gamma_a$. (Because we pick some $a \in \Gamma_x$ to construct W.) So there exists $r' \in \mathbb{Q}$ s.t $a = x + r'$. Let $r = -r'$. Then $x = a + r$.

STEP 2. We show that $W \notin \mathscr{M}$. We use proof by contradiction. Suppose that $W \in \mathscr{M}$.

case 1. (if $m(W) = 0$)

$$\infty = m(\mathbb{R}) = m\left(\bigcup_{r\in\mathbb{Q}} W_{+r}\right) \leqq \sum_{r\in\mathbb{Q}} m(W_{+r}) = 0$$

(contradiction!!)

case 2. (if $m(W) > 0$) By Theorem 2.20 Steinhaus Theorem, we $\exists\delta > 0$ s.t $(-\delta, \delta) \subset W - W$. However $(W - W) \setminus \{0\}$ are irrational numbers by the argument above. In other words, $W - W$ can not contain rational numbers, so it can not contain an interval. (An interval always contain rational numbers.) (contradiction!!)

In conclusion W is not Lebesgue measurable.

(2) Extention to the case of $\mathbb{R}^d$ is quite easy. We just need to change $\mathbb{R}, \mathbb{Q}$ into $\mathbb{R}^d, \mathbb{Q}^d$ in the discussion above.

□

55 (Additional Theorem) We show the case of $\mathbb{R}^1$. (Modification to the case of $\mathbb{R}^d$ is easy.) Let $\tilde{W}$ be a non Lebesgue measurable set on $\mathbb{R}^1$. Let $\tilde{W}_{+r} = \{x + r \mid x \in \tilde{W}\}$ where $r \in \mathbb{Q}$. Since $\bigcup_{r\in\mathbb{Q}} \tilde{W}_{+r} = \mathbb{R}^1$, we have $A = \bigcup_{r\in\mathbb{Q}} \tilde{W}_{+r} \cap A$. By sub-additivity,

$$0 < m^*(A) \leqq \sum_{r\in\mathbb{Q}} m^*(\tilde{W}_{+r} \cap A).$$

So there exists at least one $r_0 \in \mathbb{Q}$ s.t $0 < m^*(\tilde{W}_{+r_0} \cap A)$. $W \overset{\text{def}}{=} \tilde{W}_{+r_0} \cap A$ is the desired non Lebesgue measurable set.

Suppose $W \in \mathscr{M}$, by Steinhaus Theorem, $\exists\delta > 0$ s.t $(-\delta, \delta) \subset W - W = \tilde{W}_{+r_0} \cap A - \tilde{W}_{+r_0} \cap A \subset \tilde{W}_{+r_0} - \tilde{W}_{+r_0} = \tilde{W} - \tilde{W}$. However, we have already shown that $\tilde{W} - \tilde{W}$ does not contain any intervals in the previous question. So $W \notin \mathscr{M}$. □

56 (Exercise 1) We can construct a non Lebesgue measurable set $W \subset [0, 1]$ by the Additional Theorem. From each $\Gamma_x, x \in A$, we can always choose $a_x \in [0, 1] \cap \Gamma_x$. Then $W \subset [0, 1]$. Such W satisfies the given condition. □

57 (Exercise 2) We construct a non Lebesgue measurable set $W \subset [0, 1]$ using the Additional Theorem. Let $\{r_k\} \overset{\text{def}}{=} [-1, 1] \cap \mathbb{Q}$ and let $E_k \overset{\text{def}}{=} W_{+r_k}$ where $W_{+r_k} \overset{\text{def}}{=} \{w + r_k \mid w \in W\}$. Note that each E_k are disjoint. (Suppose that $E_1 \cap E_2 \neq \emptyset$. Then pick $x \in E_1 \cap E_2$. $x = w_1 + r_1 = w_2 + r_2$ where $w_1, w_2 \in W$. This implies that $w_1 - w_2 \in \mathbb{Q}$. But this can not happen.)

$$\bigcup_{k=1}^{\infty} E_k \subset [-1, 2].$$

So $m^*(\bigcup_{k=1}^{\infty} E_k) \leqq 3$ but $\sum_{k=1}^{\infty} m^*(E_k) = \infty$ because $m^*(E_k) = m^*(W_{+r_k}) = m^*(W) > 0$.
□

58 (Exercise 3) Since $(W-W)\setminus\{0\}$ does not contain any rational numbers, $\forall x \in W-W, \forall\delta>0, B(x,\delta)\not\subset(W-W)\setminus\{0\}$. ($\because B(x,\delta)$ contains rational numbers.) Therefore we conclude that $W-W$ has no interior point. □

59 (Exercise 4) We suppose that $E\Delta W \in \mathscr{M}$ and derive a contradiction. Then $E\Delta W\cap E \in \mathscr{M}$ thus $E\setminus W\in\mathscr{M}$. And $E\Delta W\setminus(E\setminus W) = W\setminus E\in\mathscr{M}$. Next $E\setminus(E\setminus W) = E\cap W\in\mathscr{M}$. Finally $W = E\cap W\cup W\setminus E\in\mathscr{M}$. (contradiction!!)

□

60 (Exercise 5) We show that $E\in\mathscr{M}\Rightarrow$

$$\sup_{F\colon \text{closed};F\subset E}\{m(F)\} = \inf_{G\colon \text{open};E\subset G}\{m(G)\}.$$

Let $S \stackrel{\text{def}}{=} \sup_{F\colon \text{closed};F\subset E}\{m(F)\}$ and $I\stackrel{\text{def}}{=}\inf_{G\colon \text{open};E\subset G}\{m(G)\}$.

case 1. $(m(E)<\infty)$ By Theorem 2.13, we have a sequence of closed sets and open sets $\{F_n\}_{n\geqq 1} : F_n\subset E,\ \{G_n\}_{n\geqq 1} : G_n\supset E$ where

$$m\left(G_n\setminus E\right)<\frac{1}{2n},\ m\left(E\setminus F_n\right)<\frac{1}{2n}$$

Then

$$\begin{aligned} I-S\leqq m(G_n)-m(F_n) &= m(G_n)-m(E)+m(E)-m(F_n)\\ &\stackrel{*}{=} m(G_n\setminus E)+m(E\setminus F_n)\\ &< \frac{1}{n}\to 0\end{aligned}$$

- $(*)$ $m(E)<\infty, E\subset G_n$ so $m(G_n\setminus E)=m(G_n)-m(E)$. Similarly, $m(E\setminus F_n) = m(E)-m(F_n)$.

case 2. $(m(E)=\infty)$ It is enough for us to show that $S=\infty$. Since $\forall\epsilon>0$, $\exists F\subset E, F$: closed s.t $m(E\setminus F)<\epsilon$. $m(E) = m(E\setminus F)+m(F)$ holds. The left hand side is ∞. If $m(F)<\infty$, the equality does not hold, hence $m(F)=\infty$. So we conclude that $S\leqq m(F)=\infty$.

□

61 (Exercise 6) Let $I\subset\mathbb{R}^d$ be a non Lebesgue measurable set. We define $E_\alpha\stackrel{\text{def}}{=}\mathbb{R}^d\setminus\{\alpha\}$ (So $E_\alpha^c=\{\alpha\}$). Then

$$\bigcap_{\alpha\in I}E_\alpha = \left(\bigcup_{\alpha\in I}E_\alpha^c\right)^c = I^c\notin\mathscr{M}.$$

□

62 (Extra Exercise 1) Let

$$\Gamma_I\stackrel{\text{def}}{=}\{J\in\Gamma\mid I\cap J\neq\emptyset\}.$$

First $\bigcup_{J\in\Gamma_I} J$ is also an interval. Second $\{\Gamma_I\}_{I\in\Gamma}$ is at most countable. ($\because$ We pick $\Gamma_{I_1}, \Gamma_{I_2} \in \{\Gamma_I\}_{I\in\Gamma}$. Then $\bigcup_{J\in\Gamma_{I_1}} J$, $\bigcup_{J\in\Gamma_{I_2}} J$ are disjoint intervals. Each interval contains rational numbers and rational numbers are countable, so disjoint intervals are countable.) Finally

$$\bigcup_{I\in\Gamma} I = \bigcup_{I\in\Gamma}\bigcup_{J\in\Gamma_I} J$$

is a countable union of intervals. So it is measurable.

$\square$

63 (Extra Exercise 2) Suppose that there exists a measure zero set Z s.t $m(f(Z)) > 0$. By Extra Theorem, there exists a non measurable set $W \notin \mathscr{M}$ s.t $W \subset f(Z)$. Then $f^{-1}(W) \subset Z$ so $f^{-1}(W)$ is a measure zero set, hence measurable. By assumption $f(f^{-1})(W) = W$ is measurable. This contradicts to the fact that W is not measurable. $\square$

§ 2.6

64 (Definition 2.3) Let $\mathscr{O}^d$ be a collection of all open sets on $\mathbb{R}^d$. T is continous $\overset{\text{def}}{=} \forall G \in \mathscr{O}^d, T^{-1}(G) \overset{\text{def}}{=} \{x \in \mathbb{R}^d \mid T(x) \in G\} \in \mathscr{O}^d$. $\square$

65 (Theorem 2.21)

STEP 1. ($\Rightarrow$) Suppose $\forall G \in \mathscr{O}^d, T^{-1}(G) \overset{\text{def}}{=} \{x \in \mathbb{R}^d \mid T(x) \in G\} \in \mathscr{O}^d$. Let $x_0 \in \mathbb{R}^n$ and let $\epsilon > 0$. Since $B \overset{\text{def}}{=} B(T(x_0), \epsilon)$ is an open set,

$$T^{-1}(B) = \{x \in \mathbb{R}^d \mid T(x) \in B(T(x_0), \epsilon)\}$$

is an open set by assumption. Since $T^{-1}(B)$ is open (and $x_0 \in T^{-1}(B)$), there exists $\delta > 0$ s.t

$$B(x_0, \delta) \subset T^{-1}(B).$$

This implies that $\forall y \in B(x_0, \delta), T(y) \in B(T(x_0), \epsilon)$. $T(y) \in B(T(x_0), \epsilon)$ is equivalent to $|T(x_0) - T(y)| < \epsilon$. In conclusion,

$$\exists \delta > 0, \text{ s.t } \forall y \in B(x_0, \delta), \ |T(x_0) - T(y)| < \epsilon.$$

STEP 2. ($\Leftarrow$) Suppose that $\forall x_0 \in \mathbb{R}^d, \forall \epsilon > 0, \exists \delta > 0$ s.t $\forall y \in B(x_0, \delta), |T(x_0) - T(y)| < \epsilon$. Let G be an open set on $\mathbb{R}^d$. We prove that $T^{-1}(G)$ is open.

case 1. If $T^{-1}(G)$ is an empty set, $T^{-1}(G)$ is open so the statement holds.

case 2. If $T^{-1}(G)$ is not an empty set, we pick an arbitrary point $x_0 \in T^{-1}(G)$. (We aim to show that $\exists \delta > 0$ s.t $B(x_0, \delta) \subset T^{-1}(G)$.) Since $T(x_0) \in G$ and G is an open set,

$$\exists \epsilon > 0 \text{ s.t } B(T(x_0), \epsilon) \subset G.$$

By assumption,

$$\exists \delta > 0 \text{ s.t } \forall y \in B(x_0, \delta), T(y) \in B(T(x_0), \epsilon) \ \subset G.$$

From this fact, we find out that

$$\begin{aligned} B(x_0,\delta) &\subset \left\{y \in \mathbb{R}^d \mid T(y) \in B(T(x_0),\epsilon)\right\} \\ &\subset \left\{y \in \mathbb{R}^d \mid T(y) \in G\right\} \\ &= T^{-1}(G). \end{aligned}$$

So we conclude that $T^{-1}(G)$ is open for all $G \in \mathscr{O}^d$.

□

66 (Example 1) Let $x = (x_1, \cdots x_d) = \sum_{i=1}^d x_i e_i \in \mathbb{R}^d$ where $e_1, \cdots, e_d$ are standard basis. We define

$$\|x\| \overset{\text{def}}{=} \left(\sum_{i=1}^d |x_i|^2\right)^2.$$

Let

$$M \overset{\text{def}}{=} \left(\sum_{i=1}^d \|T(e_i)\|^2\right)^{1/2}.$$

By linearity of T, we have

$$T(x) = T\left(\sum_{i=1}^d x_i e_i\right) = \sum_{i=1}^d x_i T(e_i).$$

Then

$$\begin{aligned} \|T(x)\| &\overset{*1}{\leqq} \sum_{i=1}^d |x_i| \cdot \|T(e_i)\| \\ &\overset{*2}{\leqq} \left(\sum_{i=1}^d |x_i|^2\right)^{1/2} \left(\sum_{i=1}^d \|T(e_i)\|^2\right)^{1/2} \\ &= M\|x\|. \end{aligned}$$

- $(*1)$ triangular inequality
- $(*2)$ Cauchy Shwartz inequality

So $\|T(x) - T(y)\| = \|T(x-y)\| \leqq M\|x-y\|$. This implies that $y \to x \Rightarrow T(y) \to T(x)$. By Theorem 2.21, T is continuous. □

67 (Theorem 2.22)

STEP 1. Let us consider an arbitrary open cover of $T(K)$ as below.

$$T(K) \subset \bigcup_{\alpha \in I} G_\alpha,\ \{G_\alpha\}_{\alpha \in I} \subset \mathscr{O}^d,$$

where $\mathscr{O}^d$ is a collection of all open sets on $\mathbb{R}^d$. By Lemma 1.20 Lindelof's covering lemma, we can always find a sub cover with countable number of open sets. Therefore we may assume that

$$T(K) \subset \bigcup_{n=1}^{\infty} G_n$$

We aim to prove that we can find a finite number $N \in \mathbb{N}$ s.t

$$T(K) \subset \bigcup_{n=1}^{N} G_n.$$

STEP 2.

$$\begin{aligned} K &\overset{*1}{\subset} T^{-1} \circ T(K) \\ &\subset T^{-1}\left(\bigcup_{n=1}^{\infty} G_n\right) \\ &\overset{*2}{=} \bigcup_{n=1}^{\infty} T^{-1}(G_n). \end{aligned}$$

- ($*1$) by definition $T^{-1} \circ T(K) = \{x \in \mathbb{R}^d \mid T(x) \in T(K)\}$ and obviously K is contained in it.
- ($*2$) generally $f^{-1}(\bigcup_{\alpha \in A} A_\alpha) = \bigcup_{\alpha \in A} f^{-1}(A_\alpha)$ holds.

Since K is a compact set, by Heine-Borel's covering theorem, we can find a finite number $N < \infty$ s.t

$$K \subset \bigcup_{n=1}^{N} T^{-1}(G_n).$$

Therefore,

$$\begin{aligned} T(K) &\subset T\left(\bigcup_{n=1}^{N} T^{-1}(G_n)\right) \\ &\overset{*3}{=} \bigcup_{n=1}^{N} T \circ T^{-1}(G_n) \\ &\overset{*4}{\subset} \bigcup_{n=1}^{N} G_n. \end{aligned}$$

- ($*3$) generally $f(\bigcup_{\alpha \in A} A_\alpha) = \bigcup_{\alpha \in A} f(A_\alpha)$ holds.
- ($*4$) by definition $T \circ T^{-1}(G_n) = \{T(x) \mid x \in T^{-1}(G_n)\} = \{T(x) \mid x \in \{y \in \mathbb{R}^d \mid T(y) \in G_n\}\} \subset G_n$.

For all open covers of $T(K)$ with countable open sets, we can always find a sub cover with finite number of open sets. So $T(K)$ is compact.

□

68 (Corollary 2.23; 2.24)

(1)

$$E = \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} F_{n,m}$$

where $F_{n,m} = F_n \cap \overline{B}(0, m)$. Then $F_{n,m}$ is a bounded and closed (= compact) set.

$$T(E) = \bigcup_{n=1}^{\infty} T(F_n) = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} T(F_{n,m})$$

Since $T(F_{n,m})$ is also a compact set (= bounded and closed) by Theorem 2.22, $T(E)$ is a countable union of closed sets. So we conclude that $T(E)$ is a F_σ set.

(2)

$$E = K \cup Z.$$

where K is a F_σ set and Z is a measure zero set. Since

$$T(E) = T(K \cup Z) = T(K) \cup T(Z),$$

and $T(K)$ is also a F_σ (by the previous result) set and $T(Z)$ is a measure zero set, $T(E)$ is measurable.

(3) Give a counter example. Let $C \subset [0, 1]$ be a Cantor set and let $\Phi(x)$ be Cantor function. $\Phi(x)$ is continuous and $m(C) = 0$. However $m(\Phi(C)) = 1$.

□

69 (Extra Theorem: Lipschitz Continuous)

(1) Let $T : \mathbb{R}^d \mapsto \mathbb{R}^d$. If there exists a positive number L s.t

$$\forall x, y \in \mathbb{R}^d, \|T(x) - T(y)\| \leqq L\|x - y\|,$$

where $\|a\| \overset{\text{def}}{=} \left(\sum_{i=1}^{d} a_i^2\right)^{1/2}$. Then we say that T is Lipschitz continuous.

(2) If $Z \subset \mathbb{R}^d$ is a measure zero set and T is Lipschitz continuous, then $T(Z)$ is also a measure zero set.

If Z is a measure zero set, for any positive number $\epsilon > 0$, we can find a countably many open balls $B_i \overset{\text{def}}{=} B(x_i, r_i)$ s.t

$$Z \subset \bigcup_{i=1}^{\infty} B_i, \ \sum_{i=1}^{\infty} m^*(B_i) < \epsilon. \ (*)$$

Let us consider the diameter of $T(B)$ where B is an open ball with radius r. Since

$$\begin{aligned}\operatorname{diam}(T(B)) &= \sup_{x,y\in B}\|T(x)-T(y)\| \\ &\leqq \sup_{x,y\in B} L\|x-y\| \\ &= L\cdot\operatorname{diam}(B) = 2Lr,\end{aligned}$$

we can cover $T(B)$ with an open ball with radius $2Lr$. Therefore

$$T(Z)\subset T\left(\bigcup_{i=1}^{\infty} B_i\right)=\bigcup_{i=1}^{\infty} T(B_i)$$

and

$$m^*(T(Z))\leqq \sum_{i=1}^{\infty} m^*\left(T(B_i)\right) < (2L)^d\cdot\epsilon.$$

by taking $\epsilon \searrow +0$, we have the desired result. So our main task in this question is to prove $(*)$.

STEP 1. First, we prove the following fact. Let $\epsilon > 0$ be an arbitrary positive number. If $Z\subset\mathbb{R}^d$ is a measure zero set, then there exists countably many open rectangles $\{I_n\}$ s.t

$$Z\subset\bigcup_{n=1}^{\infty} I_n,\ \sum_{n=1}^{\infty}|I_n|<\epsilon,$$

with

$$\max_{i=1,2\cdots,d}(b_i^{(n)}-a_i^{(n)})\leqq \min_{i=1,2\cdots,d} 2(b_i^{(n)}-a_i^{(n)}).$$

where

$$I_n \overset{\text{def}}{=} \prod_{i=1}^{d}(a_i^{(n)},b_i^{(b)}).$$

Let $\lambda\in(1,2)$. By the definition of outer measure, there exists coutably many open rectangles $\{J_n\}_{n=1}^{\infty}$ s.t

$$Z\subset\bigcup_{n=1}^{\infty} J_n,\ \sum_{n=1}^{\infty}|J_n|<\frac{\epsilon}{\lambda^d}.$$

Next, we can divide each open rectangles J_n into $\{J_{n,m}\}_{m=1}^{k_n}$ so that the longest edge length is equal or less than the twice of the shortest edge length. And we rename $\{J_{n,m}\}_{n,m}$ to $\{\tilde{I}_n\}$ by reindexing them. Since

$$|J_n| \overset{*1}{=} \sum_{m=1}^{k_n}|J_{n,m}|,$$

- $(*1)$ this holds obviously by the definition of $|\cdot|$.

we have

$$\sum_{n=1}^{\infty}|J_n| = \sum_{n=1}^{\infty}\sum_{m=1}^{k_n}|J_{n,m}| = \sum_{n=1}^{\infty}|\tilde{I}_n|.$$

$\tilde{I}_n$ can not necessarily cover Z because the boundaries among $\{J_{n,m}\}_{m=1}^{k_n}$ have been cut out. Let I_n be an open rectangle which has the same center with $\tilde{I}_n$ and each edge length is λ times of $\tilde{I}_n$. Then $Z \subset \bigcup_{n=1}^{\infty} I_n$. And we have

$$\sum_{n=1}^{\infty}|I_n| = \lambda^d \sum_{n=1}^{\infty}|\tilde{I}_n|,$$

hence,

$$\sum_{n=1}^{\infty}|I_n| < \lambda^d \cdot \frac{\epsilon}{\lambda^d} = \epsilon.$$

$\{I_n\}$ is the desired open rectangles. The proportion of each edge length is same as $\tilde{I}_n$ so the longest edge length of each I_n is also less than or equal to the twice of the shortest edge length.

STEP 2. Next we prove the following fact. Let $I = \prod_{i=1}^{d}(a_i, b_i)$ be an arbitrary open rectangle on $\mathbb{R}^d$ with

$$\max_{i=1,2\cdots d}(b_i - a_i) \leqq 2 \min_{i=1,2\cdots d}(b_i - a_i).$$

Then we can always find an open ball B s.t

$$I \subset B, m^*(B) \leqq C \cdot |I|,$$

where C is a constant which is not related to I.

Let $\ell \overset{\text{def}}{=} \min_{i=1,2\cdots,d}(b_i - a_i)$, let $r \overset{\text{def}}{=} \text{diam}(I)$ and let $C \overset{\text{def}}{=} \frac{(4\pi d)^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$.

$$\begin{aligned} r \overset{\text{def}}{=} \text{diam}(I) &= \left(\sum_{i=1}^{d}(b_i - a_i)^2\right)^{1/2} \\ &\leqq \left(d \cdot \left(\max_{i=1,2\cdots d}(b_i - a_i)\right)^2\right)^{1/2} \\ &\leqq \left(d \cdot (2\ell)^2\right)^{1/2} = 2\sqrt{d}\ell \end{aligned}$$

Moreover,

$$|I| \overset{\text{def}}{=} \prod_{i=1}^{d}(b_i - a_i) \geqq \prod_{i=1}^{d}\ell = \ell^d.$$

An open ball B with r can cover I. The outer measure of B is

$$\begin{aligned}
m^*(B) &= \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} r^d \\
&\leqq \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(2\sqrt{d}\ell\right)^d \\
&= \frac{(4\pi d)^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \ell^d \\
&\leqq \frac{(4\pi d)^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} |I| = C \cdot |I|.
\end{aligned}$$

Now the proof is complete.

STEP 3. (proof of $(*)$) Let $\epsilon > 0$ be an arbitrary positive number. Let $Z \subset \mathbb{R}^d$ be a measure zero set. For $\epsilon^* = \frac{\epsilon}{C} > 0$, there exists countably many open rectangles $\{I_n\}$ s.t

$$Z \subset \bigcup_{n=1}^{\infty} I_n, \ \sum_{n=1}^{\infty} |I_n| < \epsilon^* = \frac{\epsilon}{C}.$$

By the previous result, for each I_n, we have B_n with

$$I_n \subset B_n, \ m^*(B_n) \leqq C \cdot |I_n|$$

Therefore

$$Z \subset \bigcup_{n=1}^{\infty} B_n, \ \sum_{n=1}^{\infty} m^*(B_n) \leqq \sum_{n=1}^{\infty} C \cdot |I_n| < C \cdot \frac{\epsilon}{C}.$$

□

70 (Theorem 2.25) □

71 (Extra Exercise 1) Let $E_k \stackrel{\text{def}}{=} E \cap B(0,k)$. For all $x, y \in E_k$, we have

$$|f(x) - f(y)| \leqq e^{|x|+|y|} |x-y| \leqq e^{2k} |x-y|.$$

So $f(x)$ is Lipschitz continuous on E_k. Therefore, if $m(E) = 0$, then $m(E_k) = 0$ ($\because E_k$ is a subset of E) so $m(f(E_k)) = 0$ by Extra Theorem. Therefore

$$\begin{aligned}
m(f(E)) &= m\left(f\left(\bigcup_{n=1}^{\infty} E_k\right)\right) \\
&= m\left(\bigcup_{n=1}^{\infty} f(E_k)\right) \\
&\leqq \sum_{n=1}^{\infty} m(f(E_k)) = 0.
\end{aligned}$$

□

72 (Extra Exercise 2) Let T be a rotation on $\mathbb{R}^2$. Then

$$T(x,y) \overset{\text{def}}{=} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

So T is a linear transformation and the determinant is 1. By Theorem 2.25, we have the desired conclusion. $\square$

§ 2.7

73 (Lemma) We just have to confirm the following three conditions.

STEP 1. Since $f^{-1}(\emptyset) = \emptyset \in \Gamma$, $\emptyset \in \mathscr{A}$.

STEP 2. Suppose $A \in \mathscr{A}$, then $f^{-1}(A) \in \Gamma$. Since Γ is a $\sigma-$algebra, so $(f^{-1}(A))^c \in \Gamma$. Therefore $(f^{-1}(A))^c = f^{-1}(A^c) \in \Gamma$. This implies that $A^c \in \mathscr{A}$.

STEP 3. Let $\{A_n\}_{n=1}^{\infty} \subset \mathscr{A}$. Then for each $n = 1, 2\cdots$, $f^{-1}(A_n) \in \Gamma$. Since Γ is a $\sigma-$algebra, we have $\bigcup_{n=1}^{\infty} f^{-1}(A_n) \in \mathscr{A}$. $\bigcup_{n=1}^{\infty} f^{-1}(A_n) = f^{-1}(\bigcup_{n=1}^{\infty} A_n) \in \Gamma$. This implies that $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$.

We conclude that $\mathscr{A}$ is a σ-algebra.

$\square$

74 (Corollary) In the previous lemma, let $\Gamma \overset{\text{def}}{=} \mathscr{B}$: the family of Borel sets on $\mathbb{R}$ Then

$$\mathscr{A} \overset{\text{def}}{=} \{A \subset \mathbb{R} \mid f^{-1}(A) \in \mathscr{B}\}$$

is a $\sigma-$algebra. Moreover $\forall G \in \mathscr{O}$ (the family of open set on $\mathbb{R}$). $f^{-1}(G) \in \mathscr{O} \subset \mathscr{B}$ because f is a continuous function. This implies that $\mathscr{O} \subset \mathscr{A}$. Since $\mathscr{B}$ is the smallest $\sigma-$algebra that contains $\mathscr{O}$, so $\mathscr{B} \subset \mathscr{A}$. (because $\mathscr{A}$ is also a $\sigma-$algebra that contains $\mathscr{O}$.) Now pick $B \in \mathscr{B}$. Then $B \in \mathscr{A}$ so $f^{-1}(B) \in \mathscr{B}$ according to the definition of $\mathscr{A}$. So the proof is complete. $\square$

75 (Example: non-Borel set) Until now, we have already shown that there exists a non Lebesgue measurable set. So

$$\mathscr{M} \neq 2^{\mathbb{R}^d} \text{ where } 2^{\mathbb{R}^d} = \{B \subset \mathbb{R}^d\}.$$

We have also shown that a Borel set (or Borel-measurable) set $B \in \mathscr{B}$ is Lebesgue measurable $B \in \mathscr{M}$. Therefore

$$\mathscr{B} \subset \mathscr{M}.$$

It is natural for us to have such a question.

$$\mathscr{B} = \mathscr{M} \text{ or } \mathscr{B} \neq \mathscr{M}\ ?$$

To prove that $\mathscr{B} \neq \mathscr{M}$, we construct a set $A \in \mathscr{M}$ but $A \notin \mathscr{B}$. Let $\Phi(x)$ be the Cantor function. Let us recall that $\Phi : [0,1] \mapsto [0,1]$ and Φ is continuous on $[0,1]$. Let C be a Cantor set defined on $[0,1]$.

$$\text{i.e } C \overset{\text{def}}{=} \bigcap_{n=1} C_n \text{ and } C_n \overset{\text{def}}{=} [0,1] \setminus \bigcup_{j=1}^{n} \bigcup_{k=1}^{2^j-1} I_{j,k}$$

where $I_{1,1}=\left(\frac{1}{3},\frac{2}{3}\right), I_{2,1}=\left(\frac{1}{9},\frac{2}{9}\right), I_{2,2}=\left(\frac{5}{9},\frac{2}{3}\right)\cdots$. Note that

$$C=[0,1]\setminus\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n-1}I_{n,k},$$

and $\Phi(x)$ is a constant on each interval $x\in I_{n,k}$. So $\Phi(x)$ increases only on $x\in C$.

STEP 1. Let

$$\Psi(x)\overset{\text{def}}{=}\frac{1}{2}(x+\Phi(x))\ x\in[0,1].$$

Since x is continuous and strictly monotone increasing and $\Phi(x)$ is also continuous monotone increasing, $\Psi(x)$ is continuous and strictly monotone increasing. So Φ is a one to one mapping from $[0,1]$ to $[0,1]$. $(\Psi(0)=0,\Psi(1)=1)$. We show that

$$m\left(\Psi\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n-1}I_{n,k}\right)\right)=\frac{1}{2}.$$

Since $I_{n,k}$ are disjoint and $\Psi(x)$ is strictly monotone increasing, $\{\Psi\left(I_{n,k}\right)\}_{n,k}$ are also disjoint with each other. So we have

$$\begin{aligned}m\left(\Psi\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n-1}I_{n,k}\right)\right)&\overset{*1}{=}m\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n-1}\Psi\left(I_{n,k}\right)\right)\\&\overset{*2}{=}\sum_{n=1}^{\infty}\sum_{k=1}^{2^n-1}m\left(\Psi\left(I_{n,k}\right)\right)\end{aligned}$$

- $(*1)$ $f\left(\bigcup_{\alpha\in I}A_\alpha\right)=\bigcup_{\alpha\in I}f(A_\alpha)$.
- $(*2)$ $\{\Psi\left(I_{n,k}\right)\}_{n,k}$ are also disjoint with each other.

Furthermore, we claim that

$$m\left(\Psi(I_{n,k})\right)=\frac{1}{2}m\left(I_{n,k}\right).$$

To prove this, let

$$I_{n,k}\overset{\text{def}}{=}(a_{n,k},b_{n,k}).$$

Since $\Psi(x)$ is continuous and strictly monotone increasing,

$$\Psi(I_{n,k})=\left(\frac{a_{n,k}+\Phi(a_{n,k})}{2},\frac{b_{n,k}+\Phi(b_{n,k})}{2}\right).$$

Recall that if $x\in I_{n,k}$, then $\Phi(x)$ is constant, so $\Phi(b_{n,k})=\Phi(a_{n,k})$. Therefore

$$m(\Psi(I_{n,k}))=\frac{1}{2}m(I_{n,k}).$$

So,

$$m\left(\Psi\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n-1}I_{n,k}\right)\right)=\sum_{n=1}^{\infty}\sum_{k=1}^{2^n-1}\frac{1}{2}\cdot m(I_{n,k})\overset{*3}{=}\frac{1}{2}.$$

- (∗3) $m(C)=0 \Rightarrow m([0,1]\setminus C)=m(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^k-1} I_{n,k})=1.$

STEP 2. Since

$$m\left(\Psi\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n-1} I_{n,k}\right)\right)=\frac{1}{2},$$

we have

$$\begin{aligned} m\left([0,1]\setminus\Psi\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n-1} I_{n,k}\right)\right) &= m\left(\Psi([0,1])\setminus\Psi\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n-1} I_{n,k}\right)\right) \\ &\overset{*4}{=} m\left(\Psi\left([0,1]\setminus\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n-1} I_{n,k}\right)\right) \\ &= m(\Psi(C))=\frac{1}{2}>0. \end{aligned}$$

- (∗4) Let $f: X \mapsto Y$ be a bijective function. If $A\subset B\subset X$, then $f(B\setminus A)=f(B)\setminus f(A)$. First we claim that $f(X\setminus A)=Y\setminus f(A)$. (This is easy.) We also claim that $A_1,A_2\subset X$, then $f(A_1\cap A_2)=f(A_1)\cap f(A_2)$. $f(A_1\cap A_2)\subset f(A_1)\cap f(A_2)$ is obvious. We prove $f(A_1)\cap f(A_2)\subset f(A_1\cap A_2)$. Let $y\in f(A_1)\cap f(A_2)$. Then $\exists x_1\in A_1, x_2\in A_2$ s.t $y=f(x_1)=f(x_2)$. However, f is one to one, so $x_1=x_2$. Let $x\overset{\text{def}}{=}x_1=x_2$. Then $x\in A_1\cap A_2$. So $y\in f(A_1\cap A_2)$. Finally let $A_1=B$, $A_2=A^c$.

By the additional theorem in §2.5, there exists a non Lebesgue measurable set W with

$$W\subset\Psi(C).$$

Let

$$A\overset{\text{def}}{=}\Psi^{-1}(W).$$

We claim that A is the desired set. Note that

$$A=\Psi^{-1}(W)\subset\Psi^{-1}\circ\Psi(C)\overset{*5}{=}C$$

- (∗5) Ψ is a one to one function.

This implies that A is a measure zero set. Therefore A is Lebesgue measurable. (i.e $A\in\mathscr{M}$). However, $A\notin\mathscr{B}$. To prove this, suppose $A\in\mathscr{B}$, and we apply the previous lemma. Let $f=\Psi^{-1}$. Note that Ψ is strictly monotone increasing so Ψ^{-1} is strictly monotone increasing and continuous. Then

$$f^{-1}(A)\in\mathscr{B}.$$

However it follows that

$$f^{-1}(A)=\Psi(A)=\Psi\circ\Psi^{-1}(W)=W.$$

So $W\in\mathscr{B}$. (contradiction!!)

□

§ 2.8

76 (Exercise 1) We show that $\forall(a,b) \subset \mathbb{R}^1$, we have $m(E \cap (a,b)) = 0$. Then $\lim_{k\to\infty} m(E \cap (-n,n)) = m(E) = 0$.

STEP 1. Let $(a,b) \subset \mathbb{R}^1$ be an open interval. By assumption, we have open intervals $\{I_n\}_{n=1}^{\infty}$ s.t $E \cap (a,b) \subset \bigcup_{n=1}^{\infty} I_n$ and

$$\sum_{n=1}^{\infty} m(I_n) < (b-a)q.$$

Now we apply the assumption to each open interval I_n $(n = 1, 2\cdots)$. Then we have open intervals $\{I_{n,m}\}_{m=1}^{\infty}$ s.t $E \cap I_n \subset \bigcup_{m=1}^{\infty} I_{n,m}$ and

$$\sum_{m=1}^{\infty} m(I_{n,m}) < m(I_n)q.$$

Here $E \cap (a,b) \subset \bigcup_{n=1}^{\infty} I_n \Rightarrow E \cap (a,b) \subset \bigcup_{n=1}^{\infty} E \cap I_n$. By monotonicity and sub-additivity of Lebesgue measure, we have

$$\begin{aligned} m(E \cap (a,b)) &\leqq \sum_{n=1}^{\infty} m(E \cap I_n) \\ &\leqq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m(I_{n,m}) \\ &< \sum_{n=1}^{\infty} m(I_n)q < (b-a)q^2. \end{aligned}$$

STEP 2. Similarly, we apply the assumption to each $I_{n,m}$. We have open intervals $\{I_{n,m,k}\}_{k=1}^{\infty}$ s.t $E \cap I_{n,m} \subset \bigcup_{k=1}^{\infty} I_{n,m,k}$ and

$$\sum_{k=1}^{\infty} m(I_{n,m,k}) < m(I_{n,m,k})q.$$

Since

$$\begin{aligned} E \cap (a,b) &\subset \bigcup_{n=1}^{\infty} E \cap I_n \\ &\subset \bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty} E \cap I_{n,m} \\ &\subset \bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcup_{k=1}^{\infty} I_{n,m,k}, \end{aligned}$$

(In Step2, we have $E \cap I_n \subset \bigcup_{m=1}^{\infty} I_{n,m}$ so $E \cap I_n \subset \bigcup_{m=1}^{\infty} E \cap I_{n,m}$.) we have,

$$\begin{aligned} m(E \cap (a,b)) &\leqq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} m(I_{n,m,k}) \\ &< \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q m(I_{n,m}) < (b-a)q^3 \ (\because \text{ Step2}) \end{aligned}$$

STEP 3. We repeat the similar argument. We have $m(E \cap (a,b)) < (b-a)q^k$ for all $k = 1, 2 \cdots$ so $m(E \cap (a,b)) = 0$. ($\because 0 < q < 1$).

$\square$

77 (Exercise 2) There exists a G_δ set $H \supset A_2$ s.t $m^*(A_2) = m(H) < \infty$. Of course, $A_1 \subset H$ so $m^*(H \setminus A_2) \leqq m(H \setminus A_1) = m(H) - m(A_1) = m^*(A_2) - m(A_1) = 0$. So $H \setminus A_2$ is a measure zero set. ($\Rightarrow$ Lebesuge measurable) $A_2 = H \setminus (H \setminus A_2) \in \mathscr{M}$. $\square$

78 (Exercise 4) There never exists such a closed set F. Suppose F is a closed set and $F \neq [a,b]$.

STEP 1. First, we can find $x_0 \in (a,b)$ s.t $x_0 \notin F$. (Otherwise, $(a,b) \subset F$. Since F is closed, $a, b \in F$. So $F = [a,b]$ and this contradicts to the assumption.)

STEP 2. Second, suppose that $\forall \delta > 0, B(x_0, \delta) \cap F \neq \emptyset$. (Actually this assumption is false.) Then we can find a sequence $\{x_n\} \subset F; x_n \to x_0$. Since F is closed, $x_0 \in F$. However, this contradicts to the fact that $x_0 \notin F$. This implies that $\exists \delta > 0, B(x_0, \delta) \cap F = \emptyset$. So $[a,b] \setminus F \supset B(x_0, \delta)$ and hence we have $m([a,b] \setminus F) = (b-a) - m(F) \geqq 2\delta$. Now we conclude that $m(F) < b - a$.

$\square$

79 (Exercise 5) For example, let $\{r_k\} \stackrel{\text{def}}{=} [0,1] \cap \mathbb{Q}$ and let $\epsilon \in (0,1)$. Consider

$$B_k \stackrel{\text{def}}{=} \left(r_k - \frac{\epsilon}{2^{k+1}}, r_k + \frac{\epsilon}{2^{k+1}} \right).$$

Let

$$E = [0,1] \setminus \bigcup_{k=1}^{\infty} B_k.$$

E is the desired closed set. ($E \subset [0,1]$ and E does not contain any rational numbers in $[0,1]$.)

$$m(E) = 1 - m\left(\bigcup_{k=1}^{\infty} B_k \right) \geqq 1 - \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = 1 - \epsilon > 0.$$

$\square$

80 (Exercise 7) Let $E \stackrel{\text{def}}{=} \bigcup_{k=1}^{\infty} E_k$. Then $m(E) < \infty$. We use Fatou's lemma to

$$A_k \stackrel{\text{def}}{=} E \setminus E_k.$$

Then we have

$$m\left(\liminf_{k \to \infty} A_k \right) \leqq \liminf_{k \to \infty} m\left(A_k \right).$$

Since $m(E) < \infty$, we have

$$m(E) - m\left(\limsup_{k \to \infty} E_k \right) \leqq m(E) - \limsup_{k \to \infty} m(E_k).$$

$\square$

81 (Exercise 8) Since $\bigcap_{k=1}^{\infty} E_k \subset [0,1]$, we prove that

$$m\left([0,1] \setminus \bigcap_{k=1}^{\infty} E_k\right) = 0.$$

Since

$$\begin{aligned} m\left([0,1] \setminus \bigcap_{k=1}^{\infty} E_k\right) &= m\left(\bigcup_{k=1}^{\infty} [0,1] \setminus E_k\right) \\ &\leqq \sum_{k=1}^{\infty} m\left([0,1] \setminus E_k\right) = 0. \end{aligned}$$

Now the proof is complete.

□

82 (Exercise 9) We show that

$$m\left(\bigcup_{i=1}^{k} E_i^c\right) < 1.$$

By the assumption,

$$\sum_{i=1}^{k} m(E_i) = \sum_{i=1}^{k} (1 - m(E_i^c)) = k - \sum_{i=1}^{k} m(E_i^c) > k - 1.$$

So we have

$$\sum_{i=1}^{k} m(E_i^c) < 1.$$

By sub-additivity,

$$m\left(\bigcup_{i=1}^{k} E_i^c\right) \leqq \sum_{i=1}^{k} m(E_i^c) < 1.$$

□

83 (Exercise 11) This question is related to Vitalli's covering lemma (finite version).

STEP 1. G is an open set so $G \in \mathscr{M}$. By Theorem 2.13, $\forall \epsilon > 0, \exists F$: closed ($F \subset G$) s.t $m(G \setminus F) < \epsilon$.

case 1. ($m(G) < \infty$) Let $\epsilon = m(G) - \lambda$. $m(G \setminus F) = m(G) - m(F) < \epsilon$. So $m(F) > \lambda$.

case 2. ($m(G) = \infty$) $m(G \setminus F) + m(F) = m(G) = \infty$. So $m(F) = \infty > \lambda$.

So in any case, we can suppose that $m(F) > \lambda$. Now let $F_k \overset{\text{def}}{=} F \cap \overline{B}(0,k)$ ($\overline{B}$: closed ball). Then each F_k is a bounded closed set (a compact set). Since $F_k \nearrow F \Rightarrow m(F_k) \nearrow m(F) > \lambda$, we may find $k_0 \in \mathbb{N}$ s.t $m(F_{k_0}) > \lambda$. Let $K \overset{\text{def}}{=} F_{k_0}$.

STEP 2. $K \subset G = \bigcup_{\alpha \in \ell} B_\alpha$. By Heine-Borel's covering theorem, we may find a sub-cover with finite number of open sets. So we have $K \subset \bigcup_{k=1}^{m} B_{\alpha_k}$ where $\{\alpha_1 \cdots \alpha_m\} \subset I$.

First, we pick $B_1 \in \{B_{\alpha_1}, B_{\alpha_2}, \cdots B_{\alpha_m}\}$ which has the largest radius. If $\{B_{\alpha_k} \mid B_{\alpha_k} \cap B_1 = \emptyset\}_{k=1}^{k=m} = \emptyset$, then we terminate the process.

Second, we pick $B_2 \in \{B_{\alpha_k} \mid B_{\alpha_k} \cap B_1 = \emptyset\}_{k=1}^{k=m}$ which has the largest radius among them. If $\{B_{\alpha_k} \mid B_{\alpha_k} \cap \bigcup_{i=1}^{2} B_i = \emptyset\}_{k=1}^{k=m} = \emptyset$, then we terminate the process

Similarly we continue to choose $B_1, B_2 \cdots B_\ell$ ($\ell \leq m$) until $\{B_{\alpha_k} \mid B_{\alpha_k} \cap \bigcup_{i=1}^{l} B_i = \emptyset\}_{k=1}^{k=m}$ becomes empty.

STEP 3. We claim that $\bigcup_{k=1}^{\ell} 3B_k \supset \bigcup_{k=1}^{m} B_{\alpha_k}$ holds. Here $3B$ denotes an open ball with the same center with B but has three times the radius of B. (We sometimes define $\lambda E \overset{\text{def}}{=} \{\lambda x \mid x \in E\}$, however in this question, we define $3B$ is an open ball with the same center with B.)

Let B and $\tilde{B}$ be two open balls and suppose that $B \cap \tilde{B} \neq \emptyset$ and that B has a larger radius than $\tilde{B}$. Then $3B \supset \tilde{B}$. (You may imagine on $\mathbb{R}$ or $\mathbb{R}^2$.)

For any $\tilde{B} \in \{B_{\alpha_1}, \cdots, B_{\alpha_m}\}$, we can choose $B \in \{B_1 \cdots B_\ell\}$ s.t $B = \tilde{B}$ or B intersects with $\tilde{B}$ and B has a larger radius than $\tilde{B}$. (Let us recall that after we finish picking $B_1, \cdots, B_\ell$, the rest of balls $\{B_{\alpha_1}, \cdots, B_{\alpha_m}\} \setminus \{B_1, \cdots, B_\ell\}$ all intersect with $B_1, \cdots, B_\ell$.) Therefore $\bigcup_{k=1}^{\ell} 3B_k \supset \bigcup_{k=1}^{m} B_{\alpha_k}$ holds.

STEP 4. Finally $K \subset \bigcup_{k=1}^{m} B_{\alpha_k} \subset \bigcup_{k=1}^{\ell} 3B_k$. So $\lambda < m(K) \leqq \sum_{k=1}^{\ell} m(3B_k) = 3^n \sum_{k=1}^{\ell} m(B_k)$. ($B_1 \cdots B_\ell$ are disjoint.)

□

84 (Exercise 12) We use Corollary 2.16 and 2.17. Let $B \overset{\text{def}}{=} \bigcap_{k=1}^{\infty} B_k$, then $B_k \searrow B$ and $E = A \cap B$.

STEP 1. Since B_k and B are measurable, we have

$$m^*(A) = m^*(A \cap B_k) + m^*(A \cap B_k^c), \ (*1)$$

and

$$m^*(A) = m^*(A \cap B) + m^*(A \cap B^c). \ (*2)$$

STEP 2. Since $A \cap B_k^c \nearrow A \cap B^c$ and by Corollary 2.16, 2.17, we have

$$\begin{aligned} m^*(A) - \lim_{k \to \infty} m^*(A \cap B_k) &\overset{*3}{=} \lim_{k \to \infty} m^*(A \cap B_k^c) \\ &\overset{*4}{=} m^*(A \cap B^c) \\ &\overset{*5}{=} m^*(A) - m^*(A \cap B). \end{aligned}$$

- $(*3)$ by $(*1)$, the limit exists because $m^*(A \cap B_k^c)$ is monotone increasing.
- $(*4)$ Corollary 2.16, 2.17.
- $(*5)$ by $(*2)$.

Since $m^*(A) < \infty$, we can subtract it from the both sides and we have

$$\lim_{k\to\infty} m^*(A \cap B_k) = m^*(A \cap B).$$

This implies the desired result.

$\square$

85 (Exercise 13) Consider a G_δ set $G \supset E$ s.t $m(G) = m^*(E)$. $H \setminus G \subset H \setminus E$ and $H \setminus G \in \mathscr{M}$ so $H \setminus G$ is a measure zero set by assmption. $m^*(E) \leqq m(H) = m(H \setminus G) + m(H \cap G) = m(H \cap G) \leqq m(G) = m^*(E)$. So $m(H) = m^*(E)$. $\square$

86 (Exercise 14)

STEP 1. ($\Rightarrow$) By Theorem 2.13, we have G : open and F : closed ($G \supset E \supset F$) s.t

$$m(G \setminus E) < \frac{\epsilon}{2}, \text{ and } m(E \setminus F) < \frac{\epsilon}{2}.$$

So we have

$$\begin{aligned} m(G \cap F^c) &= m\,(G \setminus F) \\ &= m\,(G \setminus E \cup E \setminus F) \\ &\leqq m\,(G \setminus E) + m\,(E \setminus F) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Let $G_1 = G, G_2 = F^c$ and then we have the desired conclusion.

STEP 2. ($\Leftarrow$) We can find a sequence of $G_n \supset E \supset F_n$ (G_n : open, F_n : closed) s.t $m(G_n \setminus F_n) < \frac{1}{n}$. ($\because$ consider $G_n \leftarrow G_1$, $F_n \leftarrow G_2^c$). Let $K \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} F_n$. Then

$$m^*(E \setminus K) \leqq m(G_n \setminus F_n) < \frac{1}{n},$$

for all $n \in \mathbb{N}$. $\therefore m(E \setminus K) = 0$. So $E = K \cup E \setminus K \in \mathscr{M}$. (You can also use the converse of Theorem 2.13 to explain this part.)

$\square$

87 (Exercise 15) Suppose that $E_{+x_i} \overset{\text{def}}{=} \{y + x_i \mid y \in E\}$ $(i = 1, 2 \cdots n)$ are disjoint with each other. We have already proven that $E_{+x_i}, i = 1, 2 \cdots, n$ are also measurable and $m(E_{+x_i}) = m(E)$. So we have

$$m\left(\bigcup_{i=1}^{n} E_{+x_i}\right) = \sum_{i=1}^{n} m\left(E_{+x_i}\right) = nm(E) \geqq n\epsilon > 2. \ (*)$$

However, for each $i = 1, 2, \cdots, n$

$$E_{+x_i} \subset [0, 2],$$

so

$$\bigcup_{i=1}^{n} E_{+x_i} \subset [0,2].$$

From this fact, we have

$$m\left(\bigcup_{i=1}^{n} E_{+x_i}\right) \leqq 2.$$

This contradicts to $(*)$. This implies that $E_{+x_i}, i = 1, 2\cdots, n$ are not disjoint. In other words, there exist $i, j \in \{1, 2\cdots n\}$ s.t $E_{+x_i} \cap E_{+x_j} \neq \emptyset$. So

$$\exists y_1, y_2 \in E \text{ s.t } y_1 + x_i = y_2 + x_j.$$

Now we conclude that there exist $y_1, y_2 \in E$ and x_1, x_2 s.t

$$|y_1 - y_2| = |x_1 - x_2|.$$

□

88 (Exercise 16) We consider the contraposition of the statement. That is $\forall \epsilon > 0, \exists E \subset [0,1]; E \in \mathscr{M}; m(E) \geqq \epsilon$ s.t $W \cap E \in \mathscr{M} \Rightarrow W \in \mathscr{M}$.

Let $\epsilon_k = 1 - \frac{1}{k}$. There exists $E_k \subset [0,1]; E_k \in \mathscr{M}; m(E_k) \geqq 1 - \frac{1}{k}$ and $W \cap E_k \in \mathscr{M}$. Let $\tilde{E} \stackrel{\text{def}}{=} \bigcup_{k=1}^{\infty} E_k$. Then $1 - \frac{1}{k} \leqq m(E_k) \leqq m(\tilde{E}) \leqq 1$ for all $k = 1, 2\cdots$. So we have $m(\tilde{E}) = 1$ hence $m(\tilde{E}^c) = 0$. Finally $\bigcup_{k=1}^{\infty} W \cap E_k \in \mathscr{M} \Rightarrow W \cap \tilde{E} \in \mathscr{M}$ and $W \cap \tilde{E}^c \in \mathscr{M}$ because $\tilde{E}^c$ is a measure zero set so its subset $W \cap \tilde{E}^c$ is a measure zero set. So $W \in \mathscr{M}$.

□

89 (Extra Exercise 1) Let F be a closed set.

case 1. $(F \supset G)$ First, G contains all rational number on $\mathbb{R}^1$ so $\overline{G} = \mathbb{R}^1$. F is a closed set and $F \supset G$ implies that $F \supset \overline{G} = \mathbb{R}^1$. Hence $F = \mathbb{R}^1$. Second,

$$G\Delta F = (G \setminus F) \cup (F \setminus G) = F \setminus G.$$

So

$$m(G\Delta F) = m(F \setminus G) = m(\mathbb{R}^1 \setminus G) \stackrel{*}{=} m(\mathbb{R}^1) - m(G) = \infty.$$

- $(*)$ Since $m(G) < \infty$, such an operation is allowed.

case 2. $(F \not\supset G)$ $F \not\supset G$ implies that $G \setminus F \neq \emptyset$. Since $G \setminus F$ is an open set, if we pick $x_0 \in G \setminus F$ then there exists $\delta_0 > 0$ s.t

$$G \setminus F \supset B(x_0, \delta_0),$$

therefore,

$$m(G\Delta F) \geqq m(G \setminus F) \geqq m(B(x_0, \delta_0)) = 2\delta_0 > 0.$$

Now the proof is complete.

□

90 (Extra Exercise 2)

STEP 1. $\limsup_{n\to\infty} m(E_n) = \lim_{n\to\infty} \sup_{m\geqq n} m(E_m) = 1$. So for each k, we can find a subsequence n_k s.t $\sup_{m\geqq n_k} m(E_m) > 1 - \frac{1-\alpha}{2^k}$. And we can find $m_k \geqq n_k$ s.t $m(E_{m_k}) > 1 - \frac{1-\alpha}{2^k}$. So $m([0,1] \setminus E_{m_k}) < \frac{1-\alpha}{2^k}$.

STEP 2. $m([0,1]\setminus\bigcap_{k=1}^{\infty} E_{m_k}) = m(\bigcup_{k=1}^{\infty}[0,1]\setminus E_{m_k}) \leqq \sum_{k=1}^{\infty} m([0,1]\setminus E_{m_k}) \leqq 1-\alpha$. So we have $\alpha < m(\bigcap_{k=1}^{\infty} E_{m_k})$.

□

91 (Extra Exercise 3)

STEP 1. Let

$$f_1(x) \stackrel{\text{def}}{=} m(E \cap [0,x]),\ x \in [0,1].$$

Obviously, $f_1(0) = 0, f_1(1) = m(E)$ and $f_1(x)$ is monotone increasing. And $f_1(x)$ is continuous because

$$\begin{aligned} f_1(x+h) = m(E \cap [0,x+h]) &= m((E \cap [0,x]) \cup (E \cap [x,x+h])) \\ &\stackrel{*1}{\leqq} m(E \cap [0,x]) + m(E \cap [x,x+h]) \\ &\stackrel{*2}{\leqq} m(E \cap [0,x]) + m([x,x+h]) \\ &= f_1(x) + h, \end{aligned}$$

hence

$$0 \leqq f_1(x+h) - f_1(x) \leqq h.$$

- $(*1)$ by sub-additivity
- $(*2)$ by monotonicity (i.e $E \cap [x,x+h] \subset [x,x+h]$)

Therefore we can find $x_1 \in (0,1)$ s.t

$$f_1(x_1) = \frac{m(E)}{n}$$

by intermediate value theorem.

STEP 2. Similarly let

$$f_2(x) \stackrel{\text{def}}{=} m(E \cap [x_1,x]),\ x \in [x_1,1].$$

Obviously, $f_2(x_1) = 0$, $f_2(1) = \frac{n-1}{n} m(E)$ and $f_2(x)$ is also monotone increasing. Furthermore, $f_2(x)$ is continuous by the similar argument. We can find $x_2 \in (x_1,1)$ s.t

$$f_2(x_2) = \frac{1}{n} m(E)$$

STEP 3. We repeat the similar argument until we obtain $x_1, x_2 \cdots x_{n-1}$. And let $E_1 \stackrel{\text{def}}{=} E \cap [0,x_1)$, $E_2 \stackrel{\text{def}}{=} E \cap [x_1,x_2)$, ..., $E_n \stackrel{\text{def}}{=} E \cap [x_{n-1},1]$.

2.8.

□

92 (Extra Exercise 4) □

CHAPTER 3

Solutions

§ 3.1

1 (Definition 3.1) Let $\mathscr{M}$ be the family of Lebesgue measurable sets on $\mathbb{R}^d$. If

$$\forall t \in \mathbb{R},\ \{x \in E \mid f(x) > t\} = \{x \in E \mid f(x) \in (t, \infty]\} \in \mathscr{M},$$

then $f(x)$ is a measurable function defined on E.

In some textbooks, the definition of Lebesgue measurable function is different.

$$\forall B \in \mathscr{B}(\overline{\mathbb{R}}),\ f^{-1}(B) = \{x \in E \mid f(x) \in B\} \in \mathscr{M},$$

where $\mathscr{B}(\overline{\mathbb{R}}) \stackrel{\text{def}}{=} \sigma\,[\mathscr{J}],\ \mathscr{J} \stackrel{\text{def}}{=} \{[-\infty, b] \subset \overline{\mathbb{R}} \mid b \in \mathbb{R}\}$.

However, these two definitions above are equivalent. (You may skip the following proof.) Let $\mathscr{G}$ be a family of point sets on $\overline{\mathbb{R}}$, that is $\forall G \in \mathscr{G}, G \subset \overline{\mathbb{R}}$. We claim that $\forall G \in \mathscr{G}, f^{-1}(G) \in \mathscr{M}$ if and only if $\forall B \in \sigma[\mathscr{G}], f^{-1}(B) \in \mathscr{M}$.

First, $\Leftarrow$ hold obviously because $\forall G \in \mathscr{G},\ G \in \sigma[\mathscr{G}]$. Second, we prove $\Rightarrow$. Suppose that $\forall G \in \mathscr{G},\ f^{-1}(G) \in \mathscr{M}$. Let us consider the following family of sets.

$$\mathscr{A} \stackrel{\text{def}}{=} \left\{A \subset \overline{\mathbb{R}} \mid f^{-1}(A) \in \mathscr{M}\right\}$$

It is not difficult to prove that $\mathscr{A}$ is a $\sigma-$algebra (*). Furthermore, $\mathscr{G} \subset \mathscr{A}$ by assumption. Since $\sigma[\mathscr{G}]$ is the smallest $\sigma-$algebra containing $\mathscr{G}$, $\sigma[\mathscr{G}] \subset \mathscr{A}$ holds. Therefore $\forall B \in \sigma[\mathscr{G}],\ B \in \mathscr{A}\ (f^{-1}(B) \in \mathscr{M})$. So the proof of $\Rightarrow$ is also complete. Finally, $\forall t \in \mathbb{R},\ \{x \in E \mid f(x) > t\}\, \mathscr{M}$ if and only if $\forall t \in \mathbb{R},\ \{x \in E \mid f(x) \leqq t\} \in \mathscr{M}$ because $\mathscr{M}$ is a σ-algebra. Now the proof is complete.

Proof of $(*)$. $\emptyset \in \mathscr{A}$ because $f^{-1}(\emptyset) = \emptyset \in \mathscr{M}$. Let $A \in \mathscr{A}$. Then $f^{-1}(A) \in \mathscr{M}$ by definition of $\mathscr{A}$. Since $\mathscr{M}$ is a $\sigma-$algebra, $(f^{-1}(A))^c = f^{-1}(A^c) \in \mathscr{M}$. This implies that $A^c \in \mathscr{M}$. Finally let $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{A}$, then $f^{-1}(A_n) \in \mathscr{M}$ for all $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} f^{-1}(A_n) = f^{-1}(\bigcup_{n=1}^{\infty} A_n) \in \mathscr{M}$. So $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$.

□

2 (Theorem 3.1) We pick a countable subset $\{d_n\} \subset D$ s.t $d_n \searrow t$.

$$\{x \in E \mid f(x) > t\} = \bigcup_{n=1}^{\infty} \{x \in E \mid f(x) > d_n\}$$

By the assumption and the property of Lebesgue measurable sets, the right hand is Lebesgue measurable. So the proof is complete. □

3 (Example 1) Suppose that $f(x)$ is monotone increasing. Then $\{x \in [a,b] \mid f(x) > t\} = [a^*, b]$, $(a^*, b]$ or $\emptyset$ where $a^* \geqq a$. Therefore $\{x \in [a,b] \mid f(x) > t\} \in \mathscr{M}$. So $f(x)$ is a Lebesgue measurable function defined on $[a,b]$. Now the proof is complete. □

4 (Theorem 3.2) By the assumption (definition), $\{x \in E \mid f(x) > t\} \in \mathscr{M}$ for all $t \in \mathbb{R}$. As we have shown in Chapter 2, the family of Lesgue measurable sets $\mathscr{M}$ is a $\sigma-$algebra. We derive the following facts by the properties of $\sigma-$algebra $\mathscr{M}$.

- $A \in \mathscr{M} \Leftrightarrow A^c \in \mathscr{M}$.
- $\{A_n\} \subset \mathscr{M} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathscr{M}$, $\bigcap_{n=1}^{\infty} A_n \in \mathscr{M}$,

(1) $\{x \in E \mid f(x) \leqq t\} = \{x \in E \mid f(x) > t\}^c \in \mathscr{M}$.

(2) $\{x \in E \mid f(x) \geqq t\} = \bigcap_{n=1}^{\infty} \{x \in E \mid f(x) > t - \frac{1}{n}\} \in \mathscr{M}$

(3) $\{x \in E \mid f(x) < t\} = \{x \in E \mid f(x) \geqq t\}^c \in \mathscr{M}$. (Use the previous result.)

(4) $\{x \in E \mid f(x) \leqq t\} \cap \{x \in E \mid f(x) \geqq t\} \in \mathscr{M}$. (Use the previous result)

(5) $\{x \in E \mid f(x) < \infty\} = \bigcup_{n=1}^{\infty} \{x \in E \mid f(x) < t\} \in \mathscr{M}$

(6) $\{x \in E \mid f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in E \mid f(x) > t\} \in \mathscr{M}$.

(7) $\{x \in E \mid f(x) > -\infty\} = \bigcup_{n=1}^{\infty} \{x \in E \mid f(x) > -n\}$

(8) $\{x \in E \mid f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in E \mid f(x) < -n\}$

□

5 (Theorem 3.3)

(1) $\{x \in E_1 \cup E_2 \mid f(x) > t\} = \{x \in E_1 \mid f(x) > t\} \cup \{x \in E_2 \mid f(x) > t\} \in \mathscr{M}$ because $\{x \in E_1 \mid f(x) > t\}, \{x \in E_2 \mid f(x) > t\} \in \mathscr{M}$ by the assumption.

(2) $\{x \in A \mid f(x) > t\} = \{x \in E \mid f(x) > t\} \cap A \in \mathscr{M}$ because both $\{x \in E \mid f(x) > t\}, A \in \mathscr{M}$ by the assumption.

□

6 (Example 2)

$$\{x \in \mathbb{R}^d \mid \chi_E > t\} = \begin{cases} \emptyset & 1 \leqq t < \infty \\ E & 0 \leqq t < 1 \\ \mathbb{R}^d & t < 0 \end{cases}$$

And $\emptyset, E, \mathbb{R}^n \in \mathscr{M}$. So the proof is complete. $\square$

7 (Theorem 3.4)

(1)

case 1. $(c > 0)$ $\{x \in E \mid cf(x) > t\} = \{x \in E \mid f(x) > \frac{t}{c}\} \in \mathscr{M}$ because if we let $t_0 \overset{\text{def}}{=} \frac{t}{c}$ then the right hand side is $\{x \in E \mid f(x) > t_0\}$

case 2. $(c = 0)$ $\{x \in E \mid cf(x) > t\} = \{x \in E \mid 0 > t\} = \begin{cases} E & (t < 0) \\ \emptyset & (t \geqq 0) \end{cases}$. In any case, it is Lebesgue measurable.

case 3. $(c < 0)$ $\{x \in E \mid cf(x) > t\} = \{x \in E \mid f(x) < \frac{t}{c}\} \in \mathscr{M}$ by Theorem 3.2.

(2) Let $\{r_n\} \overset{\text{def}}{=} \mathbb{Q}$ be rational numbers. We use the fact that $\{x \in E \mid f_1(x) > f_2(x)\} = \bigcup_{n=1}^{\infty}\{x \in E \mid f_1(x) > r_n > f_2(x)\}$. (This holds because $\mathbb{Q}$ is a dense set in $\mathbb{R}$. If $f_1(x) > f_2(x)$, then there exists at least one rational number $r \in \mathbb{Q}$ s.t $f_1(x) > r > f_2(x)$.)

$$\begin{aligned}
\{x \in E \mid f(x) + g(x) > t\} &= \{x \in E \mid f(x) > t - g(x)\} \\
&= \bigcup_{n=1}^{\infty}\{x \in E \mid f(x) > r_n > t - g(x)\} \\
&= \bigcup_{n=1}^{\infty}\{x \in E \mid f(x) > r_n\} \cap \{x \in E \mid r_n > t - g(x)\} \\
&= \bigcup_{n=1}^{\infty}\{x \in E \mid f(x) > r_n\} \cap \{x \in E \mid g(x) > t - r_n\}
\end{aligned}$$

(3)

STEP 1. We show that $f(x)^2$ is also a Lebesgue measurable function on E if $f(x)$ is Lebesgue measurable on E.

case 1. $(t \geqq 0)$

$$\{x \in E \mid f(x)^2 > t\} = \{x \in E \mid f(x) > \sqrt{t}\} \cup \{x \in E \mid f(x) < -\sqrt{t}\} \in \mathscr{M}$$

case 2. $(t < 0)$ $\{x \in E \mid f(x)^2 > t\} = E \in \mathscr{M}$.

So $f(x)^2$ is Lebesgue measurable.

STEP 2. $f(x)g(x) = \frac{1}{4}\left((f(x) + g(x))^2 - (f(x) - g(x))^2\right)$. By the previous results, $f(x) + g(x), f(x) - g(x)$ are measurable hence so are $h_1(x) = (f(x) + g(x))^2, h_2(x) = (f(x) - g(x))^2$. Since h_1, h_2 are measurable so is $h_1 - h_2$ and $\frac{1}{4}(h_1 - h_2)$

$\square$

8 (Corollary 3.5) We need to check if the statement holds when $f(x), g(x) = \infty$ or $-\infty$.

(1) In extended real numbers, we assume the following rules. So the same argument holds.

- if $c > 0$, $c \cdot \infty = \infty$.
- if $c = 0$, $c \cdot \infty = 0$.
- if $c < 0$, $c \cdot \infty = -\infty$.

(2) If $(f(x), g(x)) = (\infty, -\infty), (-\infty, \infty)$, $f(x) + g(x)$ is not defined. In this question, we should assume that $f(x) + g(x)$ is defined on $\forall x \in E$. Then the method of proof is the same as the previous question.

However, actually, $f(x) + g(x)$ does not have to be defined every $x \in E$. $\{x \in E \mid f(x) = \infty\} \cap \{x \in E \mid g(x) = -\infty\}$ and $\{x \in E \mid f(x) = -\infty\} \cap \{x \in E \mid g(x) = \infty\}$ are measure zero sets, though $f(x) + g(x)$ is not defined on some points on E, we still may regard $f(x) + g(x)$ as a Lebesgue measureble function. Such $f(x) + g(x)$ is called a Lebesgue measurable function defined almost everywhere.

(3) Let $t \in \mathbb{R}$. Let $E_0 \overset{\text{def}}{=} \{\, x \in E \mid -\infty < f(x), g(x) < \infty\}$. $E_0 \in \mathscr{M}, E_0 \subset E$ so $f(x), g(x)$ are also measurable functions defined on E_0.

$$\begin{aligned}
& \{\, x \in E \mid f(x)g(x) > t \,\} \\
= \; & \{\, x \in E \mid f(x)g(x) > t \,\} \cap E_0 \cup \{\, x \in E \mid f(x)g(x) > t\} \cap E_0^c \\
= \; & \{\, x \in E_0 \mid f(x)g(x) > t \,\} \cup \{x \in E \mid f(x)g(x) > t\} \cap \{x \in E \mid f(x), g(x) = \pm\infty\} \\
= \; & \{\, x \in E_0 \mid f(x)g(x) > t \,\} \cup \{x \in E \mid f(x) = g(x) = \infty\} \cup \{x \in E \mid f(x) = g(x) = -\infty\}
\end{aligned}$$

□

9 (Theorem 3.6, Corollary 3.7) We can use the following fact to solve this question.

$$\left\{ x \in E \mid \sup_{m \geqq k}\{f_m(x)\} > t \right\} = \bigcup_{m=k}^{\infty} \{\, x \in E \mid f_m(x) > t\}$$

From this fact, we easily find out that $\sup_{m \geqq k}\{f_k(x)\}$ is a measurable function for each k.

(1) We just have to put $k = 1$ in the equation above.

$$\left\{ x \in E \mid \sup_{m \geqq 1}\{f_m(x)\} > t \right\} = \bigcup_{m=1}^{\infty} \{\, x \in E \mid f_m(x) > t\} \in \mathscr{M}$$

(2) Let us recall that $f(x)$ is measurable then $-f(x)$ is also measureble. So $-f_k(x)$ is also measurable for each k. $\inf_{k \geqq 1}\{f_k(x)\} = -\sup_{k \geqq 1}\{-f_k(x)\}$. So we may repeat the same argument.

(3) $\limsup_{k \to \infty} f_k(x) = \inf_{k \geqq 1} \sup_{m \geqq k}\{f_m(x)\}$. Let $g_k(x) \overset{\text{def}}{=} \sup_{m \geqq k}\{f_m(x)\}$. $g_k(x)$ is a measurable function for each k. Then $\limsup_{k \to \infty} f_k(x) = \inf_{k \geqq 1} g_k(x)$. By the previous result, we obtain the desired result.

(4) $\liminf_{k\to\infty} f_k(x) = -\limsup_{k\to\infty}(-f_k(x))$.

□

10 (Example 3) By Theorem 3.4, $f(x)$ is Lebesgue measurable if and only if $-f(x)$ is Lebesgue measurable. It is enough for us to show that $f^+(x)$ is Lebesgue measurable. Note that

$$\begin{aligned}\{x \in E \mid f^+(x) > t\} &= \{x \in E \mid \max\{f(x), 0\} > t\} \\ &= \{x \in E \mid f(x) > t\} \cup \{x \in E \mid 0 > t\}.\end{aligned}$$

Let $g(x) \stackrel{\text{def}}{=} 0$. $g(x)$ is a measurable function, so $\{x \in E \mid 0 > t\} \in \mathscr{M}$. Now the proof is complete. □

11 (Example 4) Let $f_n(x, y) \stackrel{\text{def}}{=} f(x, \frac{k}{n})$ if $y \in [\frac{k}{n}, \frac{k+1}{n}), (k = 0, \pm 1, \pm 2 \cdots)$. As n becomes larger, the partition $\{[\frac{k}{n}, \frac{k+1}{n})\}_{k\in\mathbb{Z}}$ will become finer. Since $f_n(x, y)$ is a continuous function for every fixed $x \in \mathbb{R}$, $f_n(x, y) \to f(x, y)$. It is enough for us to show that $f_n(x, y)$ is measurable because if f_n is measurable for all $n \in \mathbb{N}$ and $f_n \to f$ then f is measurable.

$$\begin{aligned}&\{\ (x, y) \in \mathbb{R} \times \mathbb{R} \mid f_n(x, y) > t\ \} \\ = &\bigcup_{k\in\mathbb{Z}} \left\{x \in \mathbb{R} \mid f\left(x, \frac{k}{n}\right) > t\right\} \times \left\{y \in \mathbb{R} \mid \frac{k}{n} \leqq y < \frac{k+1}{n}\right\}.\end{aligned}$$

For fixed $y \in \mathbb{R}$, $f(x, y)$ is a measurable function with respect to x. Moreover, if $A, B \subset \mathbb{R}^1$, $A \in \mathscr{M}$ and $B \in \mathscr{M}$ then $A \times B$ is measurable on $\mathbb{R}^2$, $(\in \mathscr{M}_2)$. So the proof is complete.
□

12 (Example 5) $\{\ x \in E \mid f(x) > t\} = \{\ x \in E \mid f(x) \in (t, \infty)\} = E \cap f^{-1}((t, \infty)) = E \cap G$ where $G = f^{-1}((t, \infty))$. Since f is continuous and (t, ∞) is open, so G is open. $G \in \mathscr{M}$ hence $E \cap G \in \mathscr{M}$. □

13 (Exercise 1) $\{x \in E \mid f(x) > 0\} \in \mathscr{M}$.

case 1. $(t \geqq 0)$

$$\{x \in E \mid f(x) > t\} = \{x \in E \mid f(x)^2 > t^2\} \cap \{x \in E \mid f(x) > 0\} \in \mathscr{M}.$$

case 2. $(t < 0)$ Let $t' = -t$.

$$\begin{aligned}\{x \in E \mid f(x) > t\} &= \{x \in E \mid f(x) > -t'\} \\ &= \{x \in E \mid f(x) > 0\} \cup \{x \in E \mid -t' < f(x) \leqq 0\}.\end{aligned}$$

And

$$\{x \in E \mid -t' < f(x) \leqq 0\} = \{x \in E \mid f(x) > 0\}^c \cap \{x \in E \mid f(x)^2 > t'^2\}^c.$$

So the proof is complete.

□

14 (Exercise 2) We show $g(x)$ is measurable. Since $h(x) = -\sup_{f\in\mathscr{F}}\{-f(x)\}$, we may show in the same method.

We show that $\{x \in (0,1) \mid g(x) > t\}$ is an open set. $(\in \mathscr{O} \subset \mathscr{B} \subset \mathscr{M})$. We pick $x_0 \in \{x \in (0,1) \mid g(x) > t\}$. Then $g(x_0) > t$. By the definition of $g(x)$, we can find $f \in \mathscr{F}$ s.t $f(x_0) > t$. Moreover, f is continuous, there exists $\delta > 0$ s.t $f(x) > t$ for all $x \in B(x_0, \delta)$. Therefore $B(x_0, \delta) \subset \{x \in (0,1) \mid g(x) > t\}$. Now the proof is complete. □

15 (Exercise 3) f_k converges at $x_0 \Leftrightarrow \limsup_{k\to\infty} f_k(x_0) = \liminf_{k\to\infty} f_k(x_0)$. So $A^c = \{\ x \in E \mid \limsup_{k\to\infty} f_k(x) > \liminf_{k\to\infty} f_k(x)\}$. Since both $\limsup_{k\to\infty} f_k(x)$ and $\liminf_{k\to\infty} f_k(x)$ are measurable so A^c is measurable because

$$\begin{aligned} A^c &= \bigcup_{r\in\mathbb{Q}}\{x \in E \mid \limsup_{k\to\infty} f_k(x) > r > \liminf_{k\to\infty} f_k(x)\} \\ &= \bigcup_{r\in\mathbb{Q}}\{x \in E \mid \limsup_{k\to\infty} f_k(x) > r\} \cap \{x \in E \mid r > \liminf_{k\to\infty} f_k(x)\} \end{aligned}$$

So A is measurable. □

16 (Exercise 4) Let $G \subset \mathbb{R}$ be an open set. By the result in Chapter 1, we have disjoint open intervals $\{(a_k, b_k)\}_k$ s.t

$$G = \bigcup_{k=1}^{\infty}(a_k, b_k).$$

$$\begin{aligned} E_1 &= \{\ x \in E \mid f(x) \in G\} \\ &= \left\{\ x \in E \mid f(x) \in \bigcup_{k=1}^{\infty}(a_k, b_k)\right\} \\ &= \bigcup_{k=1}^{\infty}\{\ x \in E \mid f(x) \in (a_k, b_k)\} \end{aligned}$$

and

$$\{\ x \in E \mid f(x) > a_k\} \cap \{\ x \in E \mid f(x) \geqq b_k\}^c \cap \in \mathscr{M}.$$

So $E_1 \in \mathscr{M}$.

Next $E \setminus E_2 = \{\ x \in E \mid f(x) \in F^c\} \in \mathscr{M}$ because F^c is open. So E_2 is also measurable. So the proof is complete. □

17 (Definition 3.2) Let N be a measure zero set. If $P(x)$ holds for $\forall x \in E \setminus N$, we say that $P(x)$ holds almost every $x \in E$. (or $P(x)$ a.e $x \in E$.) □

18 (Theorem 3.8) Let $N \overset{\text{def}}{=} \{\ x \in E \mid f(x) \neq g(x)\}$. N is a measure zero set, so $N \in \mathscr{M}$ and $E \setminus N \in \mathscr{M}$. First we divide $\{\ x \in E \mid g(x) > t\}$ into two parts as below.

$$\begin{aligned} & \{\ x \in E \mid g(x) > t\} \\ = & \{\ x \in E \mid g(x) > t\} \cap \{\ x \in E \mid f(x) = g(x)\} \\ \cup & \{\ x \in E \mid g(x) > t\} \cap \{\ x \in E \mid f(x) \neq g(x)\}. \end{aligned}$$

Next,

$$\{ x \in E \mid g(x) > t\} \cap \{ x \in E \mid f(x) = g(x)\} = \{ x \in E \mid f(x) > t\} \cap E \setminus N \in \mathscr{M}.$$

and

$$\{ x \in E \mid g(x) > t\} \cap \{ x \in E \mid f(x) \neq g(x)\} \subset N,$$

so this is also a measure zero set. $\in \mathscr{M}$.

□

19 (Extra Example) Since $f_n(x) \xrightarrow{\text{a.e}} f(x)$ on E, we have

$$f(x) = \limsup_{n\to\infty} f_n(x) \text{ (or } \liminf_{n\to\infty} f_n(x)) \text{ a.e } x \in E.$$

By Theorem 3.6, Corollary 3.7, the right hand side is a measurable function. Furthermore, we have the desired conclusion from Theorem 3.8. Now the proof is complete.

□

20 (Example 6) Let $A_k \stackrel{\text{def}}{=} \{ x \in A \mid \frac{1}{k} \leqq f(x) \leqq k \}$. Then $A_k \nearrow \{ x \in A \mid 0 < f(x) < \infty \}$. By the assumption $m(\{ x \in A \mid 0 < f(x) < \infty \}) = m(A)$. So $\lim_{k\to\infty} m(A_k) = m(A)$. Hence $\forall \delta \in (0, m(A))$, we have k_0 s.t $m(A_{k_0}) > m(A) - \delta$. Let $B \stackrel{\text{def}}{=} A_{k_0}$. This is the desired set. □

21 (Exercise 6)

STEP 1. In Chapter 2, we have already shown that $m^*(\{x\}) = 0$. Therefore a countable set such as $\mathbb{Q}$ (collection of all rational numbers) has measure zero.

STEP 2. Let $f(x) \stackrel{\text{def}}{=} 0$ and let $g(x) = \chi_{\mathbb{Q}\cap[a,b]}(x)$ where $\mathbb{Q}$ is a collection of all rational numbers. Then $f(x) = g(x)$ a.e $x \in [a, b]$ because $m(\mathbb{Q} \cap [a, b]) = 0$ thus $g(x) = 0$ a.e $x \in [a, b]$. However $g(x)$ is not continuous for all $x \in [a, b]$. (Let us pick arbitrary $x \in [a, b]$ and arbitrary $\delta > 0$. We can always find $x_1, x_2 \in B(x, \delta)$ s.t $x_1 \in \mathbb{Q}$ and $x_2 \notin \mathbb{Q}$. $f(x_1) = 0, f(x_2) = 1$. So $f(x)$ can not be continuous at x.)

□

22 (Exercise 7) Let

$$f(x) \stackrel{\text{def}}{=} \begin{cases} 1 & x \geqq 0 \\ 0 & x < 0 \end{cases}.$$

□

Then $f(x)$ is continuous a.e $x \in \mathbb{R}$. Let $g(x)$ be a continuous function on $\mathbb{R}$.

case 1. $(g(0) > 0)$ Since g is continuous, we have $\delta > 0$ s.t $\forall x \in (-\delta, 0)$ $g(x) > 0$. However $f(x) = 0$ when $x \in (-\delta, 0)$. So $f(x) = g(x)$ a.e $x \in \mathbb{R}$ can not hold.

case 2. $(g(0) \leqq 0)$ Since g is continuous, we have $\delta > 0$ s.t $\forall x \in (0, \delta)$ $g(x) \leqq 0$. However $f(x) = 1$ when $x \in (0, \delta)$. So $f(x) = g(x)$ a.e $x \in \mathbb{R}$ can not hold.

23 (Definition 3.3) Different books may have different definitions. However, we define the items in the following way. When we talk about the following items, we assume that $f(x) : E \mapsto \mathbb{R}$. (real-valued not extended real-valued)

(1) $f(x)$ is a simple function on $E \in \mathscr{M}$ means that $\{ f(x) \mid x \in E\}$ is a finite set.

(2) $f(x)$ is a (Lebesgue) measurable simple function means that $f(x)$ is a simple function and at the same time $f(x)$ is (Lebesgue) measurable. Suppose that $\{ f(x) \mid x \in E\} = \{a_1, a_2 \cdots a_p\}$ where $a_i \neq a_j$ if $i \neq j$. Let $E_i \stackrel{\text{def}}{=} \{ x \in E \mid f(x) = a_i\}$ $(i = 1 \cdots p)$. (Then E_i is measurable and disjoint with each other.) So without loss of generarility a measurable simple function $f(x)$ is written as

$$f(x) = \sum_{i=1}^{p} a_i \chi_{E_i}(x)$$

$$\text{where} \qquad E = \bigcup_{i=1}^{p} E_i, \ E_i \in \mathscr{M}, \ E_i \cap E_j = \emptyset \ (i \neq j)$$

(3) Suppose that $f(x)$ is a measurable simple function. Moreover each E_i is an iterval. Then $f(x)$ is called a step function.

□

24 (Theorem 3.9)

(1) We define

$$f_n(x) \stackrel{\text{def}}{=} \min\left\{n, \frac{[2^n f(x)]}{2^n}\right\}.$$

In this book, $[x]$ means the largest integer that is not greater than x. Then $f_n(x)$ is the desired non-negative measurable simple function. We also define

$$g_n(x) \stackrel{\text{def}}{=} \frac{[2^n f(x)]}{2^n}.$$

STEP 1. (proof of $f_n(x)$ is simple) Let us pay attention to the fact that

$$f_n(x) = \begin{cases} \frac{k}{2^n} & \text{if } \frac{k}{2^n} \leqq f(x) < \frac{k+1}{2^n}, \ k = 0, 1, \cdots n \cdot 2^n - 1 \\ n & \text{if } f(x) \geqq n \end{cases}$$

From this fact, we find out that $f_n(x)$ only takes $\{\frac{k}{2^n} \mid k = 0, 1, 2 \cdots n2^n - 1\} \cup \{n\}$. And we also find out that $f_n(x)$ is written as

$$f_n(x) = \sum_{k=0}^{n2^n - 1} \frac{k}{2^n} \chi_{\{\frac{k}{2^n} \leqq f(x) < \frac{k+1}{2^n}\}}(x) + n\chi_{\{f(x) \geqq n\}}(x) \ (*).$$

STEP 2. (proof of $f_n(x) \to f(x)$) Since $f_n(x) = \min\{n, g_n(x)\}$, it is enough for us to show that $g_n(x) \to f(x)$. Since

$$0 \leqq f(x) - g_n(x) \leqq \frac{1}{2^n},$$

$g_n(x) \to f(x)$ as $n \to \infty$.

STEP 3. $(f_n(x) \leqq f_{n+1}(x))$ Let us recall that $f_n(x) = \min\{n, g_n(x)\}$. Since $n < n+1$, it is enough for us to show that $g_n(x) \leqq g_{n+1}(x)$. Since $2[a] \leqq [2a]$, so $2[2^n f(x)] \leqq [2^{n+1} f(x)]$. Therefore $\frac{[2^n f(x)]}{2^n} \leqq \frac{[2^{n+1} f(x)]}{2^{n+1}}$.

STEP 4. (proof of $f_n(x)$ is Lebesgue measurable) First we prove some facts. Let $h_1(x), h_2(x)$ be measurable functions. Then $\min\{h_1(x), h_2(x)\}$ is also a measurable function because

$$\{\ x \in E \mid \min\{h_1(x), h_2(x) > t\ \} = \{\ x \in E \mid h_1(x) > t\ \} \cap \{\ x \in E \mid h_2(x) > t\ \}.$$

(Similarly, $\max\{h_1(x), h_2(x)\}$ is also measurable.)

Let $h(x)$ be a measurable function. Then $[h(x)]$ is also a measurable function because

$$\{\ x \in E \mid [h(x)] > t\ \} = \{\ x \in E \mid h(x) > [t+1]\ \}.$$

Now we prove that $f_n(x)$ is Lebesgue measurable. Let $c = 2^n$. $cf(x) = 2^n f(x)$ is measurable. By the previous result, $[cf(x)] = [2^n f(x)]$ is measurable. $[cf(x)]/c = [2^n f(x)]/2^n$ is measurable. Obviously, n (a constant function) is measurable. Again by the previous result, we conclude that $\min\{n, [cf(x)]/c\} = \min\{n, [2^n f(x)]/2^n\} = f_n(x)$ is measurable. Of course, you can also prove using $(*)$.

(2) Let

$$f^+(x) \overset{\text{def}}{=} \begin{cases} f(x) & f(x) \geqq 0 \\ 0 & f(x) < 0 \end{cases}$$

$$f^-(x) \overset{\text{def}}{=} \begin{cases} 0 & f(x) \geqq 0 \\ -f(x) & f(x) < 0 \end{cases}.$$

This is equivalent to $f^+(x) \overset{\text{def}}{=} \max\{f(x), 0\}, f^-(x) \overset{\text{def}}{=} \max\{0, -f(x)\}$. Then $f(x) = f^+(x) - f^-(x)$ and $|f(x)| = f^+(x) + f^-(x)$. Of course, $f^+(x)$ and $f^-(x)$ are Lebesgue measurable functions. Since $f^+(x)$ and $f^-(x)$ are non negative measurable functions, we may find sequences of non negative measurable simple functions $f_n^+(x)$ and $f_n^-(x)$ s.t $0 \leqq f_n^+(x) \nearrow f^+(x)$ and $0 \leqq f_n^-(x) \nearrow f^-(x)$. Then let $f_n(x) \overset{\text{def}}{=} f_n^+(x) - f_n^-(x)$. $|f_n(x)| \leqq |f(x)|$ and $f_n(x) \to f(x)$. (Note. $f_n^+(x) \to \infty$ and $f_n^-(x) \to \infty$ does not occur at the same time because one of $f^+(x), f^-(x)$ is always 0.)

(3) Suppose that $|f(x)| \leqq M, M < \infty$. When $n > M$, $f^+(x) - f_n^+(x) \leqq \frac{1}{2^n}$ and $f^-(x) - f_n^-(x) \leqq \frac{1}{2^n}$ because $f_n^+(x) \overset{\text{def}}{=} \min\{n, \frac{[2^n f^+(x)]}{2^n}\} = \frac{[2^n f^+(x)]}{2^n}$ $(\because |f(x)| \leqq M)$ hence $0 \leqq f^+(x) - f_n^+(x) \leqq \frac{1}{2^n}$. Since

$$\begin{aligned} |f(x) - f_n(x)| &= \left|f^+(x) - f^-(x) - f_n^+(x) + f_n^-(x)\right| \\ &\leqq |f^+(x) - f_n^+(x)| + |f^-(x) - f_n^-(x)| \\ &\leqq \frac{1}{2^{n-1}}\ (\forall x \in E), \end{aligned}$$

we have

$$\lim_{n\to\infty} \sup_{x\in E} |f(x) - f_n(x)| = 0.$$

$\square$

25 (Definition 3.4) In this book we define in the following way.

$$\operatorname{supp}(f) \stackrel{\text{def}}{=} \overline{\{\ x \in E \mid f(x) \neq 0\}}$$

$\square$

26 (Corollary 3.10) Let $f_n(x) = \sum_{i=1}^{p_n} a_i^{(n)} \chi_{E_i^{(n)}}(x)$. Since $\chi_{B(0,n)}(x) \to 1$ for every $x \in \mathbb{R}^d$ as $n \to \infty$, $f_n(x) \cdot \chi_{B(0,n)}(x) \to f(x)$ as $n \to \infty$.

$\tilde{f}_n(x) \stackrel{\text{def}}{=} f_n(x) \cdot \chi_{B(0,n)}(x) = \sum_{i=1}^{p_n} a_i^{(n)} \chi_{E_i^{(n)} \cap B(0,n)}(x)$ and $\tilde{f}_n \to f(x)$. $\operatorname{supp}(\tilde{f}_n(x)) \subset \overline{\bigcup_{i=1}^{p_n} E_i^{(n)} \cap B(0,n)} \subset \overline{B}(0,n)$. So the support is bounded. Therefore the support is a compact set. $\square$

§ 3.2

27 (Definition 3.5) If there exists a measure zero set $N : m(N) = 0$ and $\forall x \in E \setminus N, \lim_{k\to\infty} f_k(x) = f(x)$, then we say that $\{f_k\}_{k \geqq 1}$ converges to f almost everywhere on E. We denote $f_k \stackrel{\text{a.e}}{\longrightarrow} f$ on E or $f_k \to f$ a.e $x \in E$. $\square$

28 (Lemma 3.11) Let $\epsilon > 0$ be an arbitrary positive number.

STEP 1. Suppose that $f_k(x) \stackrel{\text{a.e}}{\longrightarrow} f(x)$ on E. Then we have

$$\#\{k \mid |f_k(x) - f(x)| \geqq \epsilon\} < \infty \text{ a.e } x \in E.$$

Equivalently,

$$\#\{k \mid x \in E_k(\epsilon)\ \} < \infty \text{ a.e } x \in E.$$

In other words, there is a measure zero set N, and if $x \in \mathbb{R}^d \setminus N$, then the number of k s.t $x \in E_k(\epsilon)$ is finite. So

$$\mathbb{R}^d \setminus N \subset \{\ x \in E \mid \#\{\ k \mid x \in E_k(\epsilon)\ \} < \infty\ \}.$$

Therefore

$$\limsup_{k\to\infty} E_k(\epsilon) \stackrel{*}{=} \{x \in E \mid \#\{\ k \mid x \in E_k(\epsilon)\} = \infty\} \subset N.$$

- $(*)$ Let us recall that $\limsup_{k\to\infty} A_k = \{x \mid x \in A_k$ for inifinitely many $k \in \mathbb{N}\}$

Now we conclude that

$$m(\limsup_{k\to\infty} E_k(\epsilon)) = 0.$$

STEP 2. By definition of lim sup for point sets, we have

$$m(\limsup_{k\to\infty} E_k(\epsilon)) = m\left(\bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} E_k(\epsilon)\right)$$

Furthermore,

$$m\left(\bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty}E_k(\epsilon)\right)=\lim_{j\to\infty}m(\bigcup_{k=j}^{\infty}E_k(\epsilon))\ (=0),$$

because

$$\bigcup_{k=j}^{\infty}E_k(\epsilon)$$

is a decreasing sequence of sets with respect to j and

$$\bigcup_{k=1}^{\infty}E_k(\epsilon)\subset E,\ m(E)<\infty.$$

So we are allowed to swap $m(\cdot)$ and lim by Corollary 2.8.

$\square$

29 (Theorem 3.12: Egorov) $f_k\xrightarrow{\text{a.u}}f\Rightarrow f_k\xrightarrow{\text{a.e}}f$ always holds. However if $m(E)<\infty$, $f_k\xrightarrow{\text{a.e}}f\Rightarrow f_k\xrightarrow{\text{a.u}}f$ holds. This is called Egorov's theorem. (Hence $\xrightarrow{\text{a.u}}\Leftrightarrow\xrightarrow{\text{a.e}}$ on E if $m(E)<\infty$.) We will explain it again in extra theorem.

STEP 1. In the previous lemma, let $\epsilon=\frac{1}{m}$ where $m\in\mathbb{N}$. Since

$$\lim_{j\to\infty}m\left(\bigcup_{k=j}^{\infty}E_k\left(\frac{1}{m}\right)\right)=0,$$

we can find a sufficiently large natural number $j(m)$ s.t

$$m\left(\bigcup_{k=j(m)}^{\infty}E_k\left(\frac{1}{m}\right)\right)<\frac{\delta}{2^m}.$$

By sub-additivity of a measure,

$$m\left(\bigcup_{m=1}^{\infty}\bigcup_{k=j(m)}^{\infty}E_k\left(\frac{1}{m}\right)\right)<\sum_{m=1}^{\infty}\frac{\delta}{2^m}=\delta.$$

Let

$$E_\delta\stackrel{\text{def}}{=}\bigcup_{m=1}^{\infty}\bigcup_{k=j(m)}^{\infty}E_k\left(\frac{1}{m}\right).$$

STEP 2. Finally, we show that $f_k\xrightarrow{u}f$ on $E\setminus E_\delta$. ($\xrightarrow{u}$: converge uniformly).

$$E\setminus E_\delta=\bigcap_{m=1}^{\infty}\bigcap_{k=j(m)}^{\infty}\left\{x\in E\mid |f_k(x)-f(x)|<\frac{1}{m}\right\}.$$

Let ϵ be an arbitrary positive number. If we take a sufficiently large $m_0 \in \mathbb{N}$ s.t $\frac{1}{m_0} < \epsilon$, then

$$\forall x \in E \setminus E_\delta \subset \bigcap_{k=j(m_0)}^{\infty} \left\{ x \in E \mid |f_k(x) - f(x)| < \frac{1}{m_0} \right\}.$$

So,

$$\sup_{x \in E \setminus E_\delta} |f_k(x) - f(x)| \leqq \frac{1}{m_0} < \epsilon, \ \forall k \geqq j(m_0)$$

In other words,

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t } \forall k \geqq N, \sup_{x \in E \setminus E_\delta} |f_k(x) - f(x)| < \epsilon.$$

So $f_k \stackrel{u}{\to} f$ on $E \setminus E_\delta$.

□

30 (Example 1)

$$x^n \to \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}, \text{as } n \to \infty.$$

So $x^n \to f(x)$ for all $x \in [0,1]$. However, since $f(x) = 0$ for all $x < 1$,

$$\lim_{x \nearrow 1} (x^n - f(x)) = 1.$$

Therefore $\sup_{x \in [0,1]} |x^n - f(x)| = 1$ and we have

$$\lim_{n \to \infty} \sup_{x \in [0,1]} |x^n - f(x)| = 1 \neq 0.$$

So x^n does not uniformly converge to $f(x)$. □

31 (Definition 3.6) Suppose that $|f(x)| < \infty$ a.e $x \in E$. (If we discuss $f_k \stackrel{m}{\to} f$, we may suppose that $|f| < \infty$ a.e $x \in E$.) If $\forall \epsilon > 0$,

$$\lim_{n \to \infty} m\left(\{ \ x \in E \mid \ |f_k(x) - f(x)| \geqq \epsilon \ \}\right) = 0,$$

then we say that f_k converges to f in measure on E. We denote it as $f_k \stackrel{m}{\to} f$ on E.

□

32 (Theorem 3.13) Let $f(x), g(x)$ be measurable functions defined on $E \in \mathscr{M}$. If $f(x) = g(x)$ a.e $x \in E$, we say that f and g are equivalent on E.

Here we suppose that $|f|, |g| < \infty$ a.e $x \in E$ because we are talking about convergence in measure. Let $\epsilon > 0$. $\{ \ x \in E \mid |f - g| > \epsilon \} = \{ \ x \in E \mid |f - f_k + f_k - g| > \epsilon \} \subset \{ \ x \in E \mid |f - f_k| + |f_k - g| > \epsilon \} \subset \{ \ x \in E \mid |f - f_k| > \frac{\epsilon}{2} \ \} \cup \{ \ x \in E \mid |f_k - g| > \frac{\epsilon}{2} \ \}$. By monotonicity and sub-additivity of a measure,

$$m(|f - g| > \epsilon) \leqq \lim_{k \to \infty} m(\{ \ x \in E \mid |f - f_k| > \frac{\epsilon}{2} \ \}) + m(\{ \ x \in E \mid |f_k - g| > \frac{\epsilon}{2} \ \}) = 0.$$

Therefore, $\forall n = 1, 2, \cdots,$

$$m\left(\{\ x \in E \mid |f-g| > \frac{1}{n}\right) = 0.$$

And we have

$$m\left(\bigcup_{n=1}^{\infty}\{\ x \in E \mid |f-g| > \frac{1}{n}\right) \leqq \sum_{n=1}^{\infty} m\left(\{\ x \in E \mid |f-g| > \frac{1}{n}\right) = 0.$$

The left hand side is $m\left(\{\ x \in E \mid |f-g| > 0\right) = 0$. This implies that $f = g$ a.e $x \in E$. □

33 (Theorem 3.14) We use Lemma 3.11. We have already shown that if $m(E) < \infty$ and $f_k(x) \overset{\text{a.e}}{\longrightarrow} f(x)$ then $\lim_{j\to\infty} m(\bigcup_{k\geqq j} E_k(\epsilon)) = 0$. Since

$$\limsup_{j\to\infty} m(E_j(\epsilon)) \leqq \lim_{j\to\infty} m\left(\bigcup_{k=j}^{\infty} E_k(\epsilon)\right) = 0$$

and the left hand side is

$$\limsup_{j\to\infty} m(\{\ x \in E \mid |f_j(x) - f(x)| \geqq \epsilon\ \}),$$

so we have $f_k(x) \overset{m}{\longrightarrow} f(x)$. □

34 (Extra Theorem: equivalent statements to $\overset{\text{a.e}}{\longrightarrow}$ and $\overset{\text{a.u}}{\longrightarrow}$)

(1)

STEP 1. ($\Rightarrow$) Let $\epsilon > 0$ be an arbitrary positive number. Since $f_k(x) \to f(x)$ a.e $x \in E$, $|f_k(x) - f(x)| \geqq \epsilon$ occurs only for finite k a.e $x \in E$. This implies that

$$\begin{aligned} & m\left(\limsup_{k\to\infty}\{x \in E \mid |f_k(x) - f(x)| \geqq \epsilon\}\right) \\ = \ & m\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{x \in E \mid |f_k(x) - f(x)| \geqq \epsilon\}\right) = 0 \end{aligned}$$

STEP 2. ($\Leftarrow$) Let $\epsilon > 0$ be an arbitrary positive number. Similarly,

$$m\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{x \in E \mid |f_k(x) - f(x)| \geqq \epsilon\}\right) = 0$$

implies that $|f_k(x) - f(x)| \geqq \epsilon$ occurs only for finite k a.e $x \in E$. In other words, at almost every $x \in E$, $|f_k(x) - f(x)| < \epsilon$ for sufficiently large k. This means that $f_k(x) \overset{\text{a.e}}{\longrightarrow} f(x)$.

(2) Let us recall that $f_k(x) \overset{\text{a.u}}{\longrightarrow} f(x)$ means that $\forall \delta > 0, \exists E_\delta \subset E; E_\delta \in \mathscr{M}; m(E_\delta) < \delta$ s.t $\lim_{k\to\infty} \sup_{x \in E\setminus E_\delta}\{\ |f_k(x) - f(x)|\ \} = 0$.

STEP 1. ($\Rightarrow$) Let $\epsilon > 0$ be an arbitrary positive number. Since $\sup_{x\in E\setminus E_\delta}\{\ |f_k(x)-f(x)|\ \} \to 0$ as $k \to \infty$, there exists $m_{\epsilon,\delta} \in \mathbb{N}$ s.t

$$\sup_{x\in E\setminus E_\delta}\{\ |f_k(x)-f(x)|\ \} < \epsilon, \forall k \geqq m_{\epsilon,\delta}.$$

So if $x \in E \setminus E_\delta \Rightarrow x \in \bigcap_{k\geqq m_{\epsilon,\delta}}^{\infty}\{\ x \in E \mid |f_k(x)-f(x)| < \epsilon\ \}$. Therefore $E \setminus E_\delta \subset \bigcap_{k\geqq m_{\epsilon,\delta}}^{\infty}\{\ x \in E \mid |f_k(x)-f(x)| < \epsilon\ \}$. By taking complement of the both sides, we have

$$\bigcup_{k\geqq m_{\epsilon,\delta}}^{\infty}\{\ x \in E \mid |f_k(x)-f(x)| \geqq \epsilon\ \} \subset E_\delta,$$

by monotonicity of an measure, we have

$$m\left(\bigcup_{k\geqq m_{\epsilon,\delta}}^{\infty}\{\ x \in E \mid |f_k(x)-f(x)| \geqq \epsilon\ \}\right) \leqq m(E_\delta) < \delta.$$

Therefore,

$$\limsup_{j\to\infty} m\left(\bigcup_{k\geqq j}^{\infty}\{\ x \in E \mid |f_k(x)-f(x)| \geqq \epsilon\ \}\right) \leqq \cdots \leqq m(E_\delta) < \delta.$$

Let us pay attention to the fact that $\bigcup_{k\geqq j}(\cdots)$ decreases as j increases. And also let us pay attention to the fact that the left hand side is not related to δ. Since we may take arbitrary small $\delta > 0$, so the left hand side is 0.

STEP 2. ($\Leftarrow$) Let $\epsilon = \frac{1}{j}$. Fist,

$$\lim_{m\to\infty} m\left(\bigcup_{k\geqq m}^{\infty}\{\ x \in E \mid |f_k(x)-f(x)| \geqq \frac{1}{j}\ \}\right) = 0.$$

This implies that we may find sufficiently large $m_j \in \mathbb{N}$, s.t

$$m\left(\bigcup_{k\geqq m_j}^{\infty}\{\ x \in E \mid |f_k(x)-f(x)| \geqq \frac{1}{j}\ \}\right) < \frac{\delta}{2^j},$$

for each $j = 1, 2\cdots$. By sub-additivity of an measure,

$$\begin{aligned}
& m\left(\bigcup_{j=1}^{\infty}\bigcup_{k\geqq m_j}^{\infty}\{\ x \in E \mid |f_k(x)-f(x)| \geqq \frac{1}{j}\ \}\right) \\
\leqq\ & \sum_{j=1}^{\infty} m\left(\bigcup_{k\geqq m_j}^{\infty}\{\ x \in E \mid |f_k(x)-f(x)| \geqq \frac{1}{j}\ \}\right) \\
<\ & \sum_{j=1}^{\infty}\frac{\delta}{2^j} = \delta.
\end{aligned}$$

Let
$$E_\delta \stackrel{\text{def}}{=} \bigcup_{j=1}^{\infty} \bigcup_{k \geqq m_j}^{\infty} \{ \, x \in E \mid |f_k(x) - f(x)| \geqq \frac{1}{j} \, \}.$$
Then its complement is
$$E \setminus E_\delta = \bigcap_{j=1}^{\infty} \bigcap_{k \geqq m_j}^{\infty} \{ \, x \in E \mid |f_k(x) - f(x)| < \frac{1}{j} \, \}.$$

Let $\epsilon > 0$ be an arbitrary positive number. Pick ${j_0}^{(\epsilon)} \in \mathbb{N}$ s.t $\frac{1}{j_0} < \epsilon$. If $x \in E \setminus E_\delta \Rightarrow x \in \bigcap_{k \geqq m_{j_0}}^{\infty} \{ \, x \in E \mid |f_k(x) - f(x)| < \frac{1}{j_0} \}$. So $\sup_{x \in E \setminus E_\delta} |f_k(x) - f(x)| \leqq \frac{1}{j_0} < \epsilon$ for all $k \geqq m_{j_0}$. In other words, $\forall \epsilon > 0$, there exists $m^{(\epsilon)} \stackrel{\text{def}}{=} m_{j_0}^{(\epsilon)}$ s.t $\forall k \geqq m^{(\epsilon)}$ $\sup_{x \in E \setminus E_\delta} |f_k(x) - f(x)| < \epsilon$. Therefore $f_k(x) \xrightarrow{u} f(x)$ on $E \setminus E_\delta$.

□

35 (Theorem 3.15) By using the extra theorem, the relationshop between $\xrightarrow{\text{a.u}}$, $\xrightarrow{\text{a.e}}$, $\xrightarrow{m}$ will be very clear.

(1) Since
$$\begin{aligned} & f_k(x) \xrightarrow{\text{a.u}} f(x) \\ \Leftrightarrow \quad & \lim_{m \to \infty} m \left(\bigcup_{k \geqq m}^{\infty} \{ \, x \in E \mid |f_k(x) - f(x)| \geqq \epsilon \, \} \right) = 0, \ \forall \epsilon > 0, \end{aligned}$$
by monotonicity, we have
$$\begin{aligned} & \lim_{m \to \infty} m \left(\{ \, x \in E \mid |f_m(x) - f(x)| \geqq \epsilon \, \} \right) \\ \leqq \quad & \lim_{m \to \infty} m \left(\bigcup_{k \geqq m}^{\infty} \{ \, x \in E \mid |f_k(x) - f(x)| \geqq \epsilon \, \} \right) = 0 \end{aligned}$$

(2) Since
$$\begin{aligned} & f_k(x) \xrightarrow{\text{a.e}} f(x) \\ \Leftrightarrow \quad & m \left(\bigcap_{m=1}^{\infty} \bigcup_{k \geqq m}^{\infty} \{ \, x \in E \mid |f_k(x) - f(x)| \geqq \epsilon \, \} \right) = 0, \ \forall \epsilon > 0, \end{aligned}$$
by monotonicity, we have
$$\begin{aligned} & m \left(\bigcap_{m=1}^{\infty} \bigcup_{k \geqq m}^{\infty} \{ \, x \in E \mid |f_k(x) - f(x)| \geqq \epsilon \, \} \right) \\ \leqq \quad & m \left(\bigcup_{k \geqq m}^{\infty} \{ \, x \in E \mid |f_k(x) - f(x)| \geqq \epsilon \, \} \right) = 0, \ \forall m \in \mathbb{N}. \end{aligned}$$
By taking $m \nearrow \infty$ in the right hand side, we have the desired result.

□

36 (Alternative Proof of Theorem 3.12) We give an alternative proof of Theorem 3.12 (Egorov). Suppose $m(E) < \infty$. Let

$$A_m \stackrel{\text{def}}{=} \bigcup_{k=m}^{\infty} \{ \ x \in E \mid |f_k(x) - f(x)| \geqq \epsilon \ \}.$$

Note that $m(A_1) < \infty$ ($\because A_1 \subset E$) and $\{A_m\}_{m=1}^{\infty}$ is a decreasing sequence of point sets. By Corollary 2.8 we may swap $\lim_{m\to\infty}$ and $m(\cdot)$. So we have

$$\lim_{m\to\infty} m(A_m) = m\left(\bigcap_{m=1}^{\infty} A_m\right).$$

Therefore,

$$\begin{aligned} & m\left(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}\{x \in E \mid |f_k(x) - f(x)| \geqq \epsilon\}\right) \\ = \ & \lim_{m\to\infty} m\left(\bigcup_{k=m}^{\infty}\{x \in E \mid |f_k(x) - f(x)| \geqq \epsilon\}\right). \end{aligned}$$

And by Theorem 3.15, we conclude that

$$f_k(x) \xrightarrow{\text{a.u}} f(x) \Leftrightarrow f_k(x) \xrightarrow{\text{a.e}} f(x), \text{ if } m(E) < \infty.$$

□

37 (Definition 3.7) We say that $\{f_k\}_{k\geqq 1}$ is a Cauchy sequence in measure if the following formula holds for all $\epsilon > 0$.

$$\lim_{j,k\to\infty} m\left(\{ \ x \in E \mid |f_k(x) - f_j(x)| \geqq \epsilon \ \}\right) = 0.$$

In other words, $\forall \epsilon > 0, \forall \delta > 0, \exists N^{(\epsilon,\delta)} \in \mathbb{N}$ s.t $\forall j, k \geqq N$,

$$m\left(\{ \ x \in E \ \mid \ |f_k(x) - f_j(x)| \geqq \epsilon \ \}\right) < \delta.$$

□

38 (Theorem 3.16) First, let $\epsilon = \delta = \frac{1}{2^i}$ in Definition 3.7.

STEP 1. By definition of a Cauchy sequence in measure, there exists $n_i \in \mathbb{N}$ s.t $\forall j, k \geqq n_i$,

$$m\left(\left\{x \in E \mid |f_k(x) - f_j(x)| \geqq \frac{1}{2^i}\right\}\right) < \frac{1}{2^i}.$$

So let $k = n_i, j = n_{i+1}$. Then

$$m\left(\left\{x \in E \mid |f_{n_i}(x) - f_{n_{i+1}}(x)| \geqq \frac{1}{2^i}\right\}\right) < \frac{1}{2^i}.$$

Let $g_i(x) \stackrel{\text{def}}{=} f_{n_i}(x)$ and let $E_i \stackrel{\text{def}}{=} \left\{x \in E \mid |g_i(x) - g_{i+1}(x)| \geqq \frac{1}{2^i}\right\}$. Then,

$$m\left(E_i\right) < \frac{1}{2^i}.$$

So

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) < 1 < \infty.$$

By Borel-Cantelli's Lemma (I) (See Chapter 2), this implies that

$$m\left(\limsup_{i\to\infty} E_i\right) = 0.$$

Let $N = \limsup_{i\to\infty} E_i$. E_i occurs only finite times at $x \in E \setminus N$. In other words, if i is sufficiently large, $|g_i(x) - g_{i+1}(x)| < \frac{1}{2^i}$ at $x \in E \setminus N$. (We may say that there exists $m_x \in \mathbb{N}$ s.t $\forall i \geqq m_x, |g_i(x) - g_{i+1}(x)| < \frac{1}{2^i}$ if $x \in E \setminus N$.) Therefore,

$$\sum_{i=1}^{\infty} |g_i(x) - g_{i+1}(x)| < \infty, \ \forall x \in E \setminus N.$$

Absolute convergence $\sum_{i=1}^{\infty} |\cdots| < \infty$ implies

$$\sum_{i=1}^{\infty} (g_{i+1}(x) - g_i(x)) \text{ converges } \forall x \in E \setminus N.$$

Since

$$g_k(x) = g_1(x) + \sum_{i=1}^{k-1} (g_{i+1}(x) - g_i(x)),$$

$g_k(x)$ converges if $x \in E \setminus N$. Now we let

$$f(x) \stackrel{\text{def}}{=} \begin{cases} \lim_{k\to\infty} g_k(x) & x \in E \setminus N \\ 0 & x \in N. \end{cases}$$

Then $f(x)$ is a measurable function. Recall that N is a measurable set, and $\limsup_{k\to\infty} g_k(x)$ and $\liminf_{k\to\infty} g_k(x)$ are measurable functions. Since $f(x) = (\limsup_{k\to\infty} g_k(x))\chi_{E\setminus N}(x)$, $f(x)$ is measurable. (Theorem 3.8 also can explain the measurability of $f(x)$.)

STEP 2. We show that $g_k(x) \xrightarrow{\text{a.u}} f(x)$. Let $\delta > 0$ be an arabitrary positive number. We may find $j \in \mathbb{N}$ s.t $\frac{1}{2^{j-1}} < \delta$. Recall that $m(E_i) < \frac{2}{i}$. By sub-additivity of an measure,

$$m\left(\bigcup_{i=j}^{\infty} E_i\right) \leqq \sum_{i=j}^{\infty} m\left(E_i\right) \leqq \frac{1}{2^{j-1}} < \delta.$$

Let $E_\delta \stackrel{\text{def}}{=} \bigcup_{i=j}^{\infty} E_i$. We may find $j \in \mathbb{N}$ s.t $\frac{1}{2^{j-1}} < \delta$. Then $m\left(\bigcup_{i=j}^{\infty} E_i\right) \leqq \sum_{i=j}^{\infty} m(E_i) < \frac{1}{2^{j-1}} < \delta$.

Since $E_\delta \supset \limsup_{i\to\infty} E_i$, we have $E \setminus E_\delta \subset E \setminus N$ by taking complement of the both sides. By the result of Step1, $\lim_{k\to\infty} g_k(x)$ converges on $E\setminus E_\delta (\subset E\setminus N)$. Let $x \in E\setminus E_\delta$. Since $\lim_{\ell\to\infty} g_\ell(x)$ converges,

$$|g_k(x) - f(x)| = |g_k(x) - \lim_{\ell\to\infty} g_\ell(x)| = \lim_{\ell\to\infty} |g_k(x) - g_\ell(x)|.$$

In the formula above,

- $g_k(x) = g_1(x) + \sum_{i=1}^{k-1} (g_{i+1}(x) - g_i(x))$.
- $g_\ell(x) = g_1(x) + \sum_{i=1}^{\ell-1} (g_{i+1}(x) - g_i(x))$

Also let us recall that $|g_{i+1}(x) - g_i(x)| < \frac{1}{2^i}, \forall i \geqq j$ because $x \in E \setminus E_\delta = \bigcap_{i\geqq j} E_i^c$. So if $k \geqq j$, we have

$$|g_k(x) - f(x)| \leqq \sum_{i=k}^{\infty} |g_{i+1}(x) - g_i(x)| \leqq \sum_{i=k}^{\infty} \frac{1}{2^i} = \frac{1}{2^{k-1}}.$$

This inequality holds for all $x \in E \setminus E_\delta$. Therefore,

$$\lim_{k\to\infty} \sup_{x\in E\setminus E_\delta} |g_k(x) - f(x)| = 0.$$

STEP 3. Finally, we show that $f_k(x) \xrightarrow{m} f(x)$. $m(\{x \in E \mid |f_k(x) - f(x)| \geqq \epsilon\}) \leqq m(\{x \in E \mid |f_k(x) - f_{n_i}(x)| \geqq \epsilon/2\}) + m(\{x \in E \mid |f_{n_i}(x) - f(x)| \geqq \epsilon/2\})$. Since $\{f_k(x)\}_{k\geqq 1}$ is a Cauchy sequence in measure, we can let $m(\{x \in E \mid |f_k(x) - f_{n_i}(x)| \geqq \epsilon/2\})$ be arbitrarily small by taking large k and i. Moreover, $g_i(x) = f_{n_i}(x) \xrightarrow{\text{a.u}} f(x)$ (by Step 2) so $f_{n_i}(x) \xrightarrow{m} f(x)$, so we can also let $m(\{x \in E \mid |f_{n_i}(x) - f(x)| \geqq \epsilon/2\})$ be arbitrarily small by taking large i. Now the proof is complete.

□

39 (Theorem 3.17)

STEP 1. (⇒) Let $\epsilon > 0$ be a arbitrary positive number. Suppose $f_k(x) \xrightarrow{m} f(x)$. For any subsequence k_i, $f_{k_i}(x) \xrightarrow{m} f(x)$. It is enough to show that we can find a subsequence k_i s.t $f_{k_i}(x) \xrightarrow{\text{a.u}} f(x)$. Since

$$\lim_{k\to\infty} m\left(\{x \in E \mid |f_k(x) - f(x)| \geqq \epsilon\}\right) = 0,$$

we can find a subsequence k_i s.t

$$m\left(\{x \in E \mid |f_{k_i}(x) - f(x)| \geqq \epsilon\}\right) < \frac{1}{2^i}.$$

Therefore

$$m\left(\bigcup_{i=m}^{\infty} \{x \in E \mid |f_{k_i}(x) - f(x)| \geqq \epsilon\}\right) < \frac{1}{2^{m-1}}.$$

Finally,

$$\lim_{m\to\infty} m\left(\bigcup_{i=m}^{\infty} \{x \in E \mid |f_{k_i}(x) - f(x)| \geqq \epsilon\}\right) = 0.$$

By the extra theorem, this implies that $f_{k_i}(x) \xrightarrow{\text{a.u}} f(x)$.

STEP 2. ($\Leftarrow$) We show the contraposition. We show if $f_k(x) \stackrel{m}{\not\to} f(x) \Rightarrow \exists k_i$ s.t $\forall k_{i_m}, f_{k_{i_m}}(x) \stackrel{\text{a.u}}{\not\to} f(x)$.

First, let us recall that $f_k(x) \stackrel{m}{\longrightarrow} f(x)$ means that

$$\forall\delta > 0, \forall\epsilon > 0, \exists N_{\delta,\epsilon} \text{ s.t } \forall k \geqq N, m(\{x \in E \mid |f_k(x) - f(x)| \geqq \epsilon\}) < \delta.$$

So its negation $f_k(x) \stackrel{m}{\not\to} f(x)$ is

$$\exists\delta > 0, \exists\epsilon > 0, \forall N, \exists k \geqq N \text{ s.t } m(\{x \in E \mid |f_k(x) - f(x)| \geqq \epsilon) \geqq \delta.$$

(Replace $\forall \to \exists$ and $\exists \to \forall$ and change the last part to the negation of the original statement.) Therefore we may find a subsequence k_i s.t $m(\{x \in E \mid |f_{k_i}(x) - f(x)| \geqq \epsilon) \geqq \delta$ for all $i \geqq 1$.

Next, let $\{k_{i_m}\}_{m\geqq 1}$ be an arbitrary further subsequence of $\{k_i\}_{i\geqq 1}$. Since

- $\bigcup_{m=m'}^{\infty}\{x \in E \mid |f_{k_{i_m}}(x) - f(x)| \geqq \epsilon\} \supset \{x \in E \mid |f_{k_{i_{m'}}}(x) - f(x)| \geqq \epsilon\}$ and
- $\{k_{i_m}\}_{m=1} \subset \{k_i\}_{i\geqq 1}$

we have

$$m\left(\bigcup_{m=m'}^{\infty}\{x \in E \mid |f_{k_{i_m}}(x) - f(x)| \geqq \epsilon\}\right) \geqq \delta.$$

By taking $\liminf_{m\to\infty}$, we have

$$\liminf_{m'\to\infty} m\left(\bigcup_{m=m'}^{\infty}\{x \in E \mid |f_{k_{i_m}}(x) - f(x)| \geqq \epsilon\}\right) \geqq \delta.$$

Therefore, $f_{k_{i_m}}(x) \stackrel{\text{a.u}}{\not\to} f(x)$. Now the proof is complete.

$\square$

40 (Exercise 1) Let us recall Theorem 3.17. Since $f_k(x) \stackrel{m}{\longrightarrow} g(x)$ on E, there exists a subsequence $\{k_\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{N}$ s.t $f_{k_\ell}(x) \stackrel{\text{a.e}}{\longrightarrow} g(x)$ on E. There exists two measure zero sets N_1, N_2 s.t

$$f_k(x) \to f(x)\ \forall x \in E \setminus N_1,\ f_{k_\ell}(x) \to g(x)\ \forall x \in E \setminus N_2.$$

Since a convergent sequence has a unique limit,

$$f(x) = g(x)\ \forall x \in E \setminus (N_1 \cup N_2).$$

Since $m(N_1 \cup N_2) = 0$, we conclude that $f(x) = g(x)$ a.e $x \in E$. $\square$

41 (Exercise 2) By Theorem 3.17, $f_k(x) \stackrel{m}{\longrightarrow} f(x) \Rightarrow \forall k_i, \exists k_{i_m}$ s.t $f_{k_{i_m}}(x) \stackrel{\text{a.u}}{\longrightarrow} f(x) \Rightarrow f_{k_{i_m}}(x) \stackrel{\text{a.e}}{\longrightarrow} f(x) \Rightarrow f^p_{k_{i_m}}(x) \stackrel{\text{a.e}}{\longrightarrow} f^p(x)$. Since $m(E) < \infty$, $\stackrel{\text{a.e}}{\longrightarrow} \Leftrightarrow \stackrel{\text{a.u}}{\longrightarrow}$. So $f^p_{k_{i_m}}(x) \stackrel{\text{a.e}}{\longrightarrow} f^p(x) \Rightarrow f^p_{k_{i_m}}(x) \stackrel{\text{a.u}}{\longrightarrow} f^p(x)$. Again by Theorem 3.17, $\forall k_i, \exists k_{i_m}$ s.t $f^p_{k_{i_m}}(x) \stackrel{\text{a.u}}{\longrightarrow} f^p(x) \Rightarrow f^p_{k_{i_m}}(x) \stackrel{m}{\longrightarrow} f_{k_{i_m}}(x)$. $\square$

42 (Exercise 3) For example let $E = \mathbb{R}$, let $f_k(x) = \frac{1}{k}$, and let $g(x) = x$. Let $\epsilon > 0$ be a arbitrary positive number. For sufficiently large k, $\frac{1}{k} < \epsilon$, so $\lim_{k\to\infty} m(\{x \in \mathbb{R} \mid |f_k(x) - 0| \geqq \epsilon\}) = 0$ However, $m(\{x \in \mathbb{R} \mid |f_k(x)g(x)| \geqq \epsilon\}) = m(\{x \in \mathbb{R} \mid |g(x)| \geqq k\epsilon\}) = m((-\infty, k\epsilon] \cup [k\epsilon, \infty)) = \infty$ for all $k = 1, 2\cdots$. So $f_k(x)g(x) \overset{m}{\nrightarrow} 0$. □

43 (Exercise 4) $\forall x \in (0, \pi)$, $\cos^n(x) \to 0$ and $m([0, \pi] \setminus (0, \pi)) = m(\{0, \pi\}) = 0$. So $\cos^n(x) \xrightarrow{\text{a.e}} 0$ on $[0, \pi]$. Since $m([0, \pi]) < \infty$, $\cos^n(x) \xrightarrow{\text{a.e}} 0 \Leftrightarrow \cos^n(x) \xrightarrow{\text{a.u}} 0 \Rightarrow \cos^n(x) \xrightarrow{m} 0$. So we conclude that $\cos^n(x) \xrightarrow{m} 0$ on $[0, \pi]$. □

44 (Exercise 5) Let $f_n(x) = \frac{1}{n}$. Then $\lim_{n\to\infty} m(\{x \in E \mid |f_n(x)| \geqq \epsilon\}) = 0$ because for any $\epsilon > 0$, when n is large enough, $\frac{1}{n} < \epsilon$. However $\frac{1}{n} > 0$ for all $n \geqq 1$ so $m(\{x \in E \mid |f_n(x)| > 0\}) = m(E)$. So $\lim_{n\to\infty} m(\{x \in E \mid |f_n(x)| > 0\}) = m(E) > 0$. □

45 (Exercise 6) By Theorem 3.17, since $f_k(x) \xrightarrow{m} 0$, we can find a subsequence k_i s.t $f_{k_i}(x) \xrightarrow{\text{a.u}} 0$. $\xrightarrow{\text{a.u}} \Rightarrow \xrightarrow{\text{a.e}}$, so $f_{k_i}(x) \xrightarrow{\text{a.e}} 0$. There exists a measure zero set N s.t $\forall x \in E \setminus N, f_{k_i}(x) \to 0$. Since $f_{k+1}(x) \leqq f_k(x)$, $f_{k_i}(x) \to 0$ implies that $f_k(x) \to 0$. Therefore $f_k(x) \to 0$ on $E \setminus N$. So we conclude that $f_k(x) \xrightarrow{\text{a.e}} 0$. □

46 (Exercise 7) We may suppose that an arbitrary positive number ϵ is in $(0, 1)$ without loss of generality. So let $\epsilon \in (0, 1)$.

(1)

$$\begin{aligned} m(\{x \in \mathbb{R}^d \mid |f_k(x) - 0| \geqq \epsilon\}) &= m(\{x \in \mathbb{R}^d \mid |\chi_{E_k}(x)| \geqq \epsilon\}) \\ &\overset{*1}{=} m(\{x \in \mathbb{R}^d \mid \chi_{E_k}(x) \geqq \epsilon\}) \\ &\overset{*2}{=} m(\{x \in \mathbb{R}^d \mid \chi_{E_k}(x) = 1\}) \\ &= m(E_k) \end{aligned}$$

- (∗1) $\chi_{E_k}(x) \geqq 0$.
- (∗2) $\chi_{E_k}(x)$ takes only 0 or 1. $\chi_{E_k}(x) > \epsilon$ $(0 < \epsilon < 1)$ occurs only $\chi_{E_k}(x) = 1$.

From this relationship, we can conclude that $f_k(x) \xrightarrow{m} 0$ if and only if $m(E_k) \to 0$.

(2) We use the extra theorem. $f_k(x) \xrightarrow{\text{a.e}} 0$ on $\mathbb{R}^d$ if and only if

$$\begin{aligned} m\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{x \in \mathbb{R}^d \mid |f_k(x) - 0| \geqq \epsilon\}\right) &= m\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{x \in \mathbb{R}^d \mid |\chi_{E_k}(x)| \geqq \epsilon\}\right) \\ &= m\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{x \in \mathbb{R}^d \mid \chi_{E_k}(x) = 1\}\right) \\ &= m\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) \\ &= m\left(\limsup_{k\to\infty} E_k\right). \end{aligned}$$

Now the proof is complete.

□

47 (Exercise 8) Similar to last previous exercise, we may suppose that $\epsilon \in (0,1)$ without loss of generality.

$$\begin{aligned}
& \lim_{k,j\to\infty} m(\{x \in \mathbb{R}^d \mid |\chi_{E_k}(x) - \chi_{E_j}(x)| > \epsilon\}) \\
\overset{*}{=} & \lim_{k,j\to\infty} m(\{x \in \mathbb{R}^d \mid |\chi_{E_k}(x) - \chi_{E_j}(x)| = 1\}) \\
= & \lim_{k,j\to\infty} m(\{x \in \mathbb{R}^d \mid \chi_{E_k}(x) = 1, \chi_{E_j}(x) = 0\} \cup \{x \in \mathbb{R}^d \mid \chi_{E_k}(x) = 0, \chi_{E_j}(x) = 1\}) \\
= & \lim_{k,j\to\infty} m((E_k \setminus E_j) \cup (E_j \setminus E_k)) \\
= & \lim_{k,j\to\infty} m(E_k \Delta E_j)
\end{aligned}$$

- $(*)$ $|\chi_{E_k}(x) - \chi_{E_j}(x)|$ can only take $0, 1$ and $\epsilon \in (0,1)$. So $|\chi_{E_k}(x) - \chi_{E_j}(x)| > \epsilon$ occurs when $|\chi_{E_k}(x) - \chi_{E_j}(x)| = 1$.

Now the proof is complete. □

48 (Exercise 9) First fix an arbitrary positive number $\epsilon > 0$. Let $E \overset{\text{def}}{=} \{x \in \mathbb{R}^1 \mid F(x) > \epsilon\}$. Then $m(E) < \infty$. Since $f_n(x) \overset{\text{a.e}}{\longrightarrow} 0$ on $\mathbb{R}^1$, $f_n(x) \overset{\text{a.e}}{\longrightarrow} 0$ on E. Since $m(E) < \infty$, $f_n(x) \overset{\text{a.e}}{\longrightarrow} 0$ on E implies $f_n(x) \overset{\text{a.u}}{\longrightarrow} 0$ on E. $f_n(x) \overset{\text{a.u}}{\longrightarrow} 0$ on E implies $f_n(x) \overset{m}{\longrightarrow} 0$ on E.

$$\begin{aligned}
m(\{x \in \mathbb{R}^1 \mid |f_n(x) - 0| > \epsilon\}) & \overset{*}{=} m(\{x \in \mathbb{R}^1 \mid |f_n(x)| > \epsilon\} \cap E) \\
& = m(\{x \in E \mid |f_n(x)| > \epsilon\})
\end{aligned}$$

- $(*)$ since $|f_n(x)| \leqq F(x)$ a.e $x \in \mathbb{R}^1$, $\{x \in \mathbb{R}^1 \mid |f_n(x)| > \epsilon\} \subset \{x \in \mathbb{R}^1 \mid F(x) > \epsilon\} = E$.

From the equality above, we conclude that $f_n(x) \overset{m}{\longrightarrow} 0$ on $\mathbb{R}^1$. □

49 (Exercise 10) Let us recall Theorem 3.17. $f_n(x) \overset{m}{\longrightarrow} f(x)$ on E implies that we can find a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ s.t $f_{n_k}(x) \overset{\text{a.u}}{\longrightarrow} f(x)$ on E. $f_{n_k}(x) \overset{\text{a.u}}{\longrightarrow} f(x)$ on E implies that $f_{n_k}(x) \overset{\text{a.e}}{\longrightarrow} f(x)$ on E. So there exists a measure zero set N s.t $f_{n_k}(x) \to f(x)$ for all $x \in E \setminus N$.

For every fixed $x \in E$, $f_n(x) \leqq f_{n+1}(x)$ so $f_n(x)$ has a limit and the limit is unique. So especially when $x \in E \setminus N$, $f_n(x)$ has the same limit with $f_{n_k}(x)$ (i.e $f(x)$). Therefore we conclude that $f_n(x) \to f(x)$ for all $x \in E \setminus N$. In other words, $f_n(x) \overset{\text{a.e}}{\longrightarrow} f(x)$ on E. Now the proof is complete. □

§ 3.3

50 (Theorem 3.18 Lusin) First we explain that we may suppose that $f(x)$ is real-valued (finite) without loss of generality. Let $N \overset{\text{def}}{=} \{x \in E \mid |f(x)| = \infty\}$. By assumption $m(N) = 0$. Let $\tilde{E} \overset{\text{def}}{=} E \setminus N \in \mathscr{M}$. $f(x)$ is a real-valued measurable function on $\tilde{E}$. We find a closed set $F \subset \tilde{E}$ on which $f(x)$ is continuous and $m(\tilde{E} \setminus F) < \delta$. Then $m(\tilde{E} \setminus F) = m(E \setminus F) < \delta$. Therefore F is the desired closed set. In conclusion, we may suppose that $f(x)$ is real-valued.

STEP 1. (simple measurable function) Let $\delta > 0$ be the given positive number. Let $f(x)$ be a measurable simple function on E. By the definition of measurable simple function,

$$f(x) = \sum_{i=1}^{p} a_i \chi_{E_i}(x)$$

where $E = \bigcup_{i=1}^{p} E_i$ and $E_i \cap E_j = \emptyset$ if $i \neq j$.

By Theorem 2.13, we have a closed set $F_i \subset E_i$ for each i s.t $m(E_i \setminus F_i) < \frac{\delta}{p}$. Let $F \stackrel{\text{def}}{=} \bigcup_{i=1}^{p} F_i$. Then F is also closed. ($\because$ finite union) $m(E \setminus F) = m(\bigcup_{i=1}^{p} E_i \setminus F_i) = \sum_{i=1}^{p} m(E_i \setminus F_i) < \sum_{i=1}^{p} \frac{\delta}{p} = \delta$.

Next we show that $f(x)$ is continuous on F. Let $\{x_n\} \subset F = \bigcup_{i=1}^{p} F_i$. ($F_i$: disjoint) and $x_n \to x_0$. Since F is closed, $x_0 \in F$. There exists $i_0 \in \{1, 2, \cdots, p\}$ s.t $x_0 \in F_{i_0}$. For sufficiently large n, $x_n \in F_{i_0}$. (Otherwise, if x_n is contained by $F_{i_1}, (i_1 \neq i_0)$ for infinitely many times, then we can find a subsequence x_{n_k} s.t $x_{n_k} \to x_0 \in F_{i_1}$. $\Rightarrow$ contradiction!!) So $f(x_n) = a_{i_0}$ for sufficiently large n. Hence $\lim_{n\to\infty} f(x_n) = a_{i_0} = f(x_0)$. So $f(x)$ is continuous on F.

STEP 2. ($f(x)$ is bounded measurable) By Theorem 3.9, we can find a subsequence of simple measurable functions $\{f_k(x)\}_{k \geqq 1}$ s.t $f_k(x) \xrightarrow{u} f(x)$ on E. $f_k(x)$ is continuous on a closed set $F_k \subset E; m(E \setminus F_k) < \frac{\delta}{2^k}$. Let $F \stackrel{\text{def}}{=} \bigcap_{k=1}^{\infty} F_k$. ($F$ is an intersection of closed sets. So F is closed.) Then $f_k(x)$ is continuous on F ($\because F \subset F_k$.) $m(E \setminus F) = m(\bigcup_{k=1}^{\infty} E \setminus F_k) \leqq \sum_{k=1}^{\infty} m(E \setminus F_k) < \delta$. Since $f_k(x) \xrightarrow{u} f(x)$ on E hence $f_k(x) \xrightarrow{u} f(x)$ on F. A sequence of continuous function uniformly converges to $f(x)$, so $f(x)$ is continuous on F.

STEP 3. (general case) Let $g(x) \stackrel{\text{def}}{=} \frac{f(x)}{1+|f(x)|} \in (-1, 1)$. Since $g(x)$ is bounded, we can find a closed set $F \subset E; m(E \setminus F) < \delta$ s.t $g(x)$ is continuous on F. Since $f(x) = \frac{g(x)}{1-|g(x)|}$, $f(x)$ is also continuous on F.

□

51 (Corollary 3.19)

(1) By Theorem 3.18 Lusin's theorem, we can find a closed set $F \subset E; m(E \setminus F) < \delta$ s.t $f(x)$ is continuous on F. By Theorem 1.27 (or Tieze Extension theorem), there exists a continuous function $g(x) \in C(\mathbb{R}^d)$ s.t $f(x) = g(x)$ on F. So we have

$$m(\{x \in E \mid |f(x) - g(x)| > 0\}) \leqq m(E \setminus F) < \delta.$$

In Theorem 1.27, we proved that if $|f(x)| \leqq M$ on F (F : closed set) and $f(x)$ is continuous on F then we can find $g(x) \in C(\mathbb{R}^d)$ s.t $g(x) = f(x)$ on F and $|g(x)| \leqq M$ on $\mathbb{R}^d$. So if $f(x)$ is bounded, then $g(x)$ is also bounded.

(2) By the previous result, we have $\tilde{g}(x) \in C(\mathbb{R}^d)$ s.t

$$m(\{x \in E \mid f(x) \neq \tilde{g}(x)\}) < \delta.$$

(There is a closed set $F \subset \{x \in E \mid f(x) = \tilde{g}(x)\}$ with $m(E \setminus F) < \delta$.) However $\tilde{g}(x)$ does not necessarily have a compact support. So let us find a continuous function $\phi(x) \in C(\mathbb{R}^d)$ with

$$\phi(x) \stackrel{\text{def}}{=} \begin{cases} 0 & x \notin B(0,r) \supset E \\ 1 & x \in F \end{cases}.$$

Then $g(x) \stackrel{\text{def}}{=} \tilde{g}(x) \cdot \phi(x)$ is a desired function.

Since E is bounded, we can find $n \in \mathbb{N}$ s.t $E \subset B(0,n)$ and let $r = n+1$. For example,

$$\phi(x) \stackrel{\text{def}}{=} \max\{0, 1 - \text{dist}(x,F)\}.$$

By Theorem 1.25, $\text{dist}(x,F)$ is continuous on $\mathbb{R}^d$. So $\phi(x)$ is also continuous on $\mathbb{R}^d$. $\text{dist}(x,F) = 0$ if $x \in F$. Let $x \notin B(0,r)$. There exists $y \in F$ s.t $|x-y| = \text{dist}(x,F)$. By triangular inequality, $|x-y| \geqq |x| - |y|$. Note that $|x| \geqq r = n+1$ and $|y| \leqq n$. So $|x-y| \geqq 1$. So $\phi(x) = 0$.

□

52 (Corollary 3.20) Let $\{\delta_k\}_{k \geqq 1}$ be a sequence of positive numbers s.t $\delta \searrow 0$ as $k \to \infty$. We can find a closed set $F_k \subset E; m(E \setminus F_k) < \delta_k$ s.t $f(x)$ is continuous on F_k. By Tieze Extension theorem, we can find $g_k(x) \in C(\mathbb{R}^n)$ s.t $f(x) = g_k(x)$ on F_k. ($g_k(x)$ is continuous so $g_k(x)$ is measurable.) Let $\epsilon > 0$ be an arbitrary positive number. Since

$$\begin{aligned} m(\{x \in E \mid |f(x) - g_k(x)| \geqq \epsilon\}) &\leqq m(\{x \in E \mid |f(x) - g_k(x)| > 0\}) \\ &\leqq m(E \setminus F_k) < \delta_k, \end{aligned}$$

$g_k(x) \xrightarrow{m} f(x)$. $g_k(x) \xrightarrow{m} f(x) \Rightarrow \exists\{k_m\}_{m \geqq 1}$ s.t $g_{k_m}(x) \xrightarrow{\text{a.u}} f(x) \Rightarrow g_{k_m}(x) \xrightarrow{\text{a.e}} f(x)$. $\tilde{g}_m(x) \stackrel{\text{def}}{=} g_{k_m}(x)$ is the desired sequence of continuous functions on $\mathbb{R}^n$.

□

53 (Example 1)

STEP 1. $f(x+y) = f(x) + f(y)$ implies that $f(x+h) - f(x) = f(h)$. If $f(x)$ is continuous at $x = 0$, then $|f(x+h) - f(x)| = |f(h)| \to 0$ as $h \to 0$, so we can conclude that $f(x)$ is continuous on $\mathbb{R}$. So we prove that $f(x)$ is continuous at $x = 0$.

STEP 2. $f(x)$ is Lebesgue measurable on $\mathbb{R}$ so $f(x)$ is measurable on $[-M,M]$. ($M > 0$). This is because $\{x \in [-M,M] \mid f(x) > t\} = \{x \in \mathbb{R} \mid f(x) > t\} \cap [-M,M] \in \mathscr{M}$. By Lusin's theorem, we can find a closed set $F \subset [-M,M]; m([-M,M] \setminus F) < \delta$ s.t $f(x)$ is continuous on F. We suppose $\delta < 2M$ then $m(F) > 0$.

Since F is a compact set and $f(x)$ is continuous on F, so $f(x)$ is uniformly continuous. Therefore, $\forall \epsilon > 0$, $\exists \delta_1 > 0$ s.t. $\forall x, y \in F; |x-y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

STEP 3. By Steinhaus' theorem, $F - F$ contains an interval $[-\delta_2, \delta_2]$ because $m(F) > 0$. Let $\delta_0 = \min\{\delta_1, \delta_2\}$. Let $h \in (-\delta_0, \delta_0)$. Since $(-\delta_0, \delta_0) \subset [-\delta_2, \delta_2] \subset F - F$, we can find $x, y \in F$ s.t $h = x - y$. $|f(h)| = |f(x-y)| = |f(x) - f(y)| < \epsilon, (\because |x - y| = |h| < \delta_0 \leqq \delta_1)$. In conclusion, $\forall \epsilon > 0$, $\exists \delta_0$ s.t $\forall h \in (-\delta_0, \delta_0)$, $|f(h)| < \epsilon \Leftrightarrow f(x)$ is continuous at $x = 0$.

□

54 (Exercise 1) (This is similar to §3.1 Exercise 7.) Let $f(x) \stackrel{\text{def}}{=} \chi_{[0,\infty)}(x)$. Suppose $g(x)$ is continuous on $\mathbb{R}$.

case 1. $(g(0) > 0)$ There exists $(-\delta, 0), (\delta > 0)$ s.t $\forall x \in (-\delta, 0)$, $g(x) < 0$. So $m(\{x \in \mathbb{R} \mid |f(x) - g(x)| > 0\}) \geqq m((-\delta, 0)) = \delta > 0$.

case 2. $(g(0) \leqq 0)$ $g(0) < 1$. There exists $(0, \delta), (\delta > 0)$ s.t $\forall x \in (0, \delta)$, $g(x) < 1$. So $m(\{x \in \mathbb{R} \mid |f(x) - g(x)| > 0\}) \geqq m((0, \delta)) = \delta > 0$.

So we conclude that there does not exist $g \in C(\mathbb{R})$ s.t $m(\{x \in \mathbb{R} \mid |f(x) - g(x)| > 0\}) = 0$.

□

55 (Exercise 2) Let $\epsilon > 0$ be an arbitrary positive number and let us fix ϵ.

STEP 1. By Corollary 3.19, we have a sequence of $g_n(x) \in C(\mathbb{R}^1)$ s.t

$$m(\{x \in [a,b] \mid f(x) \neq g_n(x)\}) < \frac{1}{n}.$$

STEP 2. Since $g_n(x) \in C(\mathbb{R}^1)$, $g_n(x) \in C([a,b])$. We apply Weierstrass's approximation theorem. There exists a polynomial $P_n(x)$ s.t

$$|g_n(x) - P_n(x)| < \epsilon, \ \forall x \in [a,b].$$

STEP 3.

$$\begin{aligned}
& m(\{x \in [a,b] \mid |f(x) - P_n(x)| > \epsilon\}) \\
= \ & m(\{x \in [a,b] \mid |f(x) - g_n(x) + g_n(x) + P_n(x)| > \epsilon\}) \\
\leqq \ & m(\{x \in [a,b] \mid |f(x) - g_n(x)| > 0\} \cup \{x \in [a,b] \mid |g_n(x) - P_n(x)| > \epsilon\}) \\
\leqq \ & m(\{x \in [a,b] \mid |f(x) - g_n(x)| > 0\}) + m(\{x \in [a,b] \mid |g_n(x) - P_n(x)| > \epsilon\}) \\
= \ & m(\{x \in [a,b] \mid f(x) \neq g_n(x)\}) + 0 < \frac{1}{n}.
\end{aligned}$$

So we have

$$P_n(x) \xrightarrow{m} f(x) \text{ on } [a,b].$$

By Theorem 3.17, we have a subsequence n_k s.t

$$P_{n_k}(x) \xrightarrow{\text{a.u}} f(x) \text{ on } [a,b].$$

Since $P_{n_k}(x) \xrightarrow{\text{a.u}} f(x)$ on $[a,b]$ implies that $P_{n_k}(x) \xrightarrow{\text{a.e}} f(x)$ on $[a,b]$ (Theorem 3.15), $\{P_{n_k}(x)\}_{k \geqq 1}$ is the desired sequence of polynomial.

□

56 (Lemma 3.21)

STEP 1. ($\Leftarrow$) Let $G = (t, \infty)$. Since $f(x)$ is real-valued (not extend real-valued), so $\{x \in \mathbb{R}^d \mid f(x) > t\} = \{x \in \mathbb{R}^n \mid t < f(x) < \infty\} = f^{-1}(G) \in \mathscr{M}$. So $f(x)$ is Lebesgue measurable.

STEP 2. ($\Rightarrow$) $G \subset \mathbb{R}, G \in \mathscr{O}$, so $\exists (a_i, b_i)$ s.t $G = \bigcup_{i=1}^{\infty}(a_i, b_i)$. (See Chapter1: Theorem 1.19.) So $f^{-1}(G) = f^{-1}(\bigcup_{i=1}^{\infty}(a_i, b_i)) = \bigcup_{i=1}^{\infty} f^{-1}((a_i, b_i)) = \bigcup_{i=1}^{\infty}\{x \in \mathbb{R}^d \mid a_i < f(x) < b_i\} = \bigcup_{i=1}^{\infty}\{x \in \mathbb{R}^d \mid f(x) > a_i\} \setminus \{x \in \mathbb{R}^d \mid f(x) \geqq b_i\} \in \mathscr{M}$.

□

57 (Supplement to Lemma 3.21)

STEP 1. ($\Leftarrow$) Let $B = (t, \infty) \in \mathscr{B}(\mathbb{R}^1)$. Since $f(x)$ is real-valued (not extend real-valued), so $\{x \in \mathbb{R}^n \mid f(x) > t\} = \{x \in \mathbb{R}^d \mid t < f(x) < \infty\} = f^{-1}(B) \in \mathscr{M}$. So $f(x)$ is Lebesgue measurable.

STEP 2. ($\Rightarrow$) Suppose that $f(x)$ is Legbesgue measurable. Let us consider the following family of sets. ($\mathscr{M}$: the family of Lebesgue measurable sets.)

$$\mathscr{A} \overset{\text{def}}{=} \{A \subset \mathbb{R} \mid f^{-1}(A) \in \mathscr{M}\}.$$

It is easy to verify that $\mathscr{A}$ is a σ-algebra. By Lemma 3.21, $\forall G \in \mathscr{O}^1, f^{-1}(G) \in \mathscr{M}$ so $G \in \mathscr{A}$. This means that $\mathscr{O}^1 \subset \mathscr{A}$. Since $\mathscr{B}(\mathbb{R}^1) \overset{\text{def}}{=} \sigma[\mathscr{O}^1]$ is the smallest $\sigma-$ algebra which contains $\mathscr{O}^1$, $\mathscr{B}(\mathbb{R}^1) \subset \mathscr{A}$. $\forall B \in \mathscr{B}(\mathbb{R}^1)$, $B \in \mathscr{A}$. In other words, $f^{-1}(B) \in \mathscr{M}$ holds for all $B \in \mathscr{B}(\mathbb{R}^1)$.

□

58 (Theorem 3.22) Let $G \overset{\text{def}}{=} (t, \infty)$. ($G \in \mathscr{O}^1$.) Then $h^{-1}(G) = g^{-1} \circ f^{-1}(G) \in \mathscr{M}$. $f(x)$ is a continuous function, so $f^{-1}(G) \in \mathscr{O}^1$. Since $g(x)$ is Lebesgue measurable, $g^{-1}(f^{-1}(G)) \in \mathscr{M}$ by Lemma 3.21. □

59 (Lemma 3.23, Corollary 3.24) By Lemma 3.21, we show that $\forall G \in \mathscr{O}^1$, $T^{-1} \circ f^{-1}(G) \in \mathscr{M}$. Since $f(x)$ is Lebesgue measurable, by Lemma 3.21, $f^{-1}(G) \in \mathscr{M}$. Let $E \overset{\text{def}}{=} f^{-1}(G)$. By Theorem 2.14, $E = H \setminus Z$ where H is a G_δ set and Z is a measure zero set. $T^{-1}(E) = T^{-1}(H \setminus Z) = T^{-1}(H) \setminus T^{-1}(Z)$. By assumption, $T^{-1}(Z)$ is also a measure zero set, so $T^{-1}(Z)$ is measurable. Let $H = \bigcap_{k=1}^{\infty} G_k$. Then $T^{-1}(H) = T^{-1}(\bigcap_{k=1}^{\infty} G_k) = \bigcap_{k=1}^{\infty} T^{-1}(G_k)$. By the definition of continuous transformation (See Chapter 2), $T^{-1}(G_k) \in \mathscr{O}^d \subset \mathscr{M}$, therefore $\bigcap_{k=1}^{\infty} T^{-1}(G_k) \in \mathscr{M}$. □

60 (Exercise 1) $f(x)^{g(x)} = \exp(\ln(f(x)^{g(x)})) = \exp(g(x)\ln(f(x)))$. Since $\ln(\cdot)$ is a continuous function, $\ln(f(x))$ is Lebesgue measurable. And $g(x)\ln(f(x))$ is Lebesgue measurable. Since $\exp(\cdot)$ is a continuous function, $\exp(g(x)\ln(f(x)))$ is Lebesgue measurable.

□

61 (Exercise 2) Since $g(x)$ is monotone increasing, $\{x \in [a,b] \mid g \circ f(x) > t\} = \{x \in [a,b] \mid f(x) \geqq u\}$ or $= \{x \in [a,b] \mid f(x) > u\}$ for some u. Since $f(x)$ is Lebesgue measurable, $\{x \in [a,b] \mid f(x) \geqq u\}, \{x \in [a,b] \mid f(x) > u\} \in \mathscr{M}$ $\square$

62 (Exercise 3) Let $\tilde{f} : \mathbb{R}^{2d} \mapsto \overline{\mathbb{R}}$ and $\tilde{f}(x,y) \stackrel{\text{def}}{=} f(x)$. Since $\{(x,y) \in \mathbb{R}^{2d} \mid \tilde{f}(x,y) > t\} = \{x \in \mathbb{R}^n \mid f(x) > t\} \times \mathbb{R}^n \in \mathscr{M}_{2d}$, $\tilde{f}(x,y)$ is a Lebesgue measurable function on $\mathbb{R}^{2d}$. (In Chapter 2, we show that $A, B \in \mathscr{M}_1$ then $A \times B \in \mathscr{M}_2$. Similarly, $A, B \in \mathscr{M}_d$ then $A \times B \in \mathscr{M}_{2d}$)

Let $T(x,y) : \mathbb{R}^{2d} \mapsto \mathbb{R}^{2d}$ and $T(x,y) \stackrel{\text{def}}{=} (x-y, x+y)$. Then $T(x,y)$ is a linear transformation. ($\because T(ax,ay) = aT(x,y)$ and $T((x_1,y_1)+(x_2,y_2)) = T(x_1,y_1)+T(x_2,y_2)$.) So $T(x,y)$ is a continuous transformation. (See §2.6 Example 1)

Finally, since $f(x-y) = \tilde{f}(T(x,y))$ and by Lemma 3.23, we conclude that $f(x-y)$ is a Lebesgue measurable function on $\mathbb{R}^{2d}$. $\square$

63 (Exercise 4)

STEP 1. We define

$$f_n(x,y) \stackrel{\text{def}}{=} n\left(\frac{m}{n} - x\right) f\left(\frac{m-1}{n}, y\right) + n\left(x - \frac{m-1}{n}\right) f\left(\frac{m}{n}, y\right),$$

if $x \in \left[\frac{m-1}{n}, \frac{m}{n}\right), (m \in \mathbb{Z})$. An equivalent definition is

$$\begin{aligned} f_n(x,y) \quad &\stackrel{\text{def}}{=} \quad \sum_{m\in\mathbb{Z}} \left(n\left(\frac{m}{n} - x\right) f\left(\frac{m-1}{n}, y\right) \right. \\ &+ \quad \left. n\left(x - \frac{m-1}{n}\right) f\left(\frac{m}{n}, y\right) \right) \cdot \chi_{[\frac{m-1}{n}, \frac{m}{n})}(x). \end{aligned}$$

STEP 2. We prove that $f_n(x,y) \to f(x,y)$ as $n \to \infty$. For each n and $x \in \mathbb{R}$, there exists $m_{n,x}$ s.t $x \in [\frac{m-1}{n}, \frac{m}{n})$. (Note that m is related to n and x). Then we have

$$\left|x - \frac{m-1}{n}\right|, \ \left|\frac{m}{n} - x\right| \leqq \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Since $f(x,y)$ is a continuous function of x (if y is fixed), we have

$$f\left(\frac{m-1}{n}, y\right), f\left(\frac{m}{n}, y\right) \to f(x,y) \text{ as } n \to \infty.$$

Also let us note that

$$n\left|\frac{m}{n} - x\right|, n\left|x - \frac{m-1}{n}\right| \leqq 1.$$

Finally,

$$\begin{aligned} & |f_n(x,y) - f(x,y)| \\ = & \left| n\left(\frac{m}{n} - x\right) f\left(\frac{m-1}{n}, y\right) + n\left(x - \frac{m-1}{n}\right) f\left(\frac{m}{n}, y\right) - f(x,y) \right| \\ = & \left| n\left(\frac{m}{n} - x\right)\left(f\left(\frac{m-1}{n}, y\right) - f(x,y)\right) + n\left(x - \frac{m-1}{n}\right)\left(f\left(\frac{m}{n}, y\right) - f(x,y)\right) \right| \\ \leqq & \ 1 \cdot \left| f\left(\frac{m-1}{n}, y\right) - f(x,y) \right| + 1 \cdot \left| f\left(\frac{m}{n}, y\right) - f(x,y) \right| \to 0 \text{ as } n \to \infty \end{aligned}$$

STEP 3. $f_n(g(y), y) \to f(g(y), y)$ as $n \to \infty$ by the previous result. It is easy to find out that $f_n(g(y), y)$ is Lebesgue measurable. So we conclude that $f(g(y), y)$ is Lebesgue measurable because it is a limit of a Lebesuge measurable function.

□

64 (Exercise 5) Let us recall that we constructed a Lebesgue measurable set which is not Borel measurable in Chapter2. Let $\Psi(x) \stackrel{\text{def}}{=} \frac{x+\Phi(x)}{2}, x \in [0,1]$ where $\Phi(x)$ is Cantor function defined on $[0,1]$.

Let us recall that $m(\Psi(C)) = 1/2$ hence $\exists W \notin \mathscr{M}; W \subset \Psi(C) \subset [0,1]$ (C : Cantor set). Since $\Psi^{-1}(W) \subset C$ and C is a measure zero set, so $\Psi^{-1}(W)$ is also a measure zero set. (Hence $\Psi^{-1}(W)$ is Lebesgue measurable.) Let $f(x) \stackrel{\text{def}}{=} \chi_{\Psi^{-1}(W)}(x)$. Then $f(x)$ is a Lebesgue measurabe function. Let $g(x) \stackrel{\text{def}}{=} \Psi^{-1}(x)$. Let us recall that $g(x)$ is a continuous function . Let us consider $f \circ g(x) = \chi_{\Psi^{-1}(W)}(\Psi^{-1}(x))$. Since $\{x \in [0,1] \mid f \circ g(x) > 0\} = W \notin \mathscr{M}$, $f \circ g(x)$ is not Lebesgue measurable.

□

§ 3.4

65 (Exercise 1) When I is not countable, $S(x)$ is not necessarily measurable. For example, let I be a non Lebesgue measurable set on $\mathbb{R}^1$. We define $f_a(x) \stackrel{\text{def}}{=} \chi_a(x)$. Then $f_a(x)$ is a Lebesgue measurable function for each $a \in I$ because $\{a\}$ is a measure zero set. However $S(x) = \chi_I(x)$ and $\{x \in \mathbb{R} \mid S(x) > t\} = I \notin \mathscr{M}$, if $0 \leqq t < 1$. □

66 (Exercise 2)

STEP 1. Let us recall that if G is an open set on $\mathbb{R}^d$ (especially $d \geqq 2$), there exist a countable number of (disjoint) half open rectangles s.t

$$G = \bigcup_{n \in \mathbb{N}} (a_{n,1}, b_{n,1}] \times \cdots \times (a_{n,d}, b_{n,d}].$$

(See Chapter 1. Theorem 1.19)

STEP 2.

$$\begin{aligned} \{x \in [a,b] \mid F(x) > t\} &= \{x \in [a,b] \mid f \circ (g_1(x), g_2(x)) > t\} \\ &= \{x \in [a,b] \mid (g_1(x), g_2(x)) \in f^{-1}(t, \infty)\}. \end{aligned}$$

Let $G = f^{-1}(t, \infty) \subset \mathbb{R}^2$. Since $f(x_1, x_2)$ is a continuous function on $\mathbb{R}^2$, G is an open set on $\mathbb{R}^2$. So

$$\begin{aligned}
& \{x \in [a,b] \mid (g_1(x), g_2(x)) \in f^{-1}(t,\infty)\} \\
= & \{x \in [a,b] \mid (g_1(x), g_2(x)) \in G)\} \\
= & \{x \in [a,b] \mid (g_1(x), g_2(x)) \in \bigcup_{n=1}^{\infty}(a_{n,1}, b_{n,1}] \times (a_{n,2}, b_{n,2}]\} \\
= & \bigcup_{n=1}^{\infty}\{x \in [a,b] \mid (g_1(x), g_2(x)) \in (a_{n,1}, b_{n,1}] \times (a_{n,2}, b_{n,2}]\} \\
= & \bigcup_{n=1}^{\infty}\{x \in [a,b] \mid g_1(x) \in (a_{n,1}, b_{n,1}]\} \cap \{x \in [a,b] \mid g_2(x) \in (a_{n,2}, b_{n,2}]\} \in \mathscr{M}
\end{aligned}$$

In the last part, note that

$$\{x \in [a,b] \mid g_1(x) \in (a_{n,1}, b_{n,1}]\} = \{x \in [a,b] \mid g_1(x) > a_{n,1}\} \setminus \{x \in [a,b] \mid g_1(x) > b_{n,1}\})$$

So the proof is complete.

□

67 (Exercise 3) Note that

$$f'_+(x) \overset{\text{def}}{=} \lim_{h \searrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{n\to\infty} \frac{f(x+1/n) - f(x)}{1/n}.$$

For each $n \in \mathbb{N}$, $\frac{f(x+1/n)-f(x)}{1/n}$ is Lebesgue measurable. By assumption, the limit exists. So $\lim_{n\to\infty} \frac{f(x+1/n)-f(x)}{1/n}$ is Lebesgue measurable. □

68 (Exercise 4) Let

$$g_n(x) \overset{\text{def}}{=} f(x)\chi_{\{|f(x)| \leqq n\}}(x).$$

Note that

$$|f(x) - g_n(x)| = |f(x)|\chi_{\{|f(x)|>n\}}(x),$$

and hence

$$\begin{aligned}
\{x \in E \mid |f(x) - g_n(x)| > 0\} &= \{x \in E \mid |f(x)|\chi_{\{|f(x)|>n\}} > 0\} \\
&= \{x \in E \mid |f(x)| > n\}.
\end{aligned}$$

Let $A_n \overset{\text{def}}{=} \{x \in E \mid |f(x)| > n\}$. Since $|f(x)| < \infty$ a.e $x \in E$,

$$m\left(\bigcap_{n=1}^{\infty} A_n\right) = m\left(\bigcap_{n=1}^{\infty}\{x \in E \mid |f(x)| > n\}\right) = m\left(\{x \in E \mid |f(x)| = \infty\}\right) = 0.$$

Moreover, $\{A_n\}_{n\geqq 1}$ is a decreasing sequence of point sets and $m(E) < \infty$ hence $m(A_1) < \infty$. Therefore

$$\lim_{n\to\infty} m(A_n) = m\left(\bigcap_{n=1}^{\infty} A_n\right) = 0.$$

This implies that there exists a sufficiently large $n_0 \in \mathbb{N}$ s.t $m(A_{n_0}) < \epsilon$. So $m(A_{n_0}) = m(\{x \in E \mid |f(x)| > n_0\}) = m(\{x \in E \mid |f(x) - g_{n_0}(x)| > 0\}) < \epsilon$. $g_{n_0}(x)$ is bounded. So this is the desired function.

We can also answer the question using Lusin's Theorem. There exists a closed set $F \subset E$ s.t $m(E \setminus F) < \epsilon$ s.t $f(x)$ is continuous on F. Let us define

$$g_\epsilon(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & x \in F \\ 0 & x \in E \setminus F \end{cases}.$$

Note that F is also bounded, so F is compact. Since $f(x)$ is continuous on a compact set F, $f(x)$ is bounded on F. Therefore $g_\epsilon(x)$ is also bounded on E. It is not difficult to verify that $g_\epsilon(x)$ is Lebesgue measurable on E. Finally,

$$\begin{aligned} m(\{x \in E \mid |f(x) - g_\epsilon(x)| > 0\}) &= m(\{x \in E \mid f(x) \neq g_\epsilon(x)\}) \\ &\leqq m(E \setminus F) < \epsilon \end{aligned}$$

□

69 (Exercise 5) By the assumption, $f_k(x) \xrightarrow{\text{a.u}} f(x)$. Let us recall that $f_k(x) \xrightarrow{\text{a.u}} f(x)$ always implies $f_k(x) \xrightarrow{\text{a.e}} f(x)$. (See Theorem 3.15, Extra Theorem.) So we conclude that $f_k(x) \xrightarrow{\text{a.e}} f(x)$. □

70 (Exercise 6)

STEP 1. Note that

$$\lim_{j\to\infty} m\left(\{x \in E \mid \sup_{k\geqq j}\{f_k(x)\} \geqq \epsilon\}\right) = 0, \ \forall\epsilon > 0, \ \cdots \ \text{(i)}$$

if and only if

$$\lim_{j\to\infty} m\left(\{x \in E \mid \sup_{k\geqq j}\{f_k(x)\} > \epsilon'\}\right) = 0, \ \forall\epsilon' > 0, \ \cdots \ \text{(ii)}$$

First we prove (i) $\Rightarrow$ (ii). Suppose that (i) holds. For all $\epsilon' > 0$, we can always take $\epsilon > 0$ s.t $0 < \epsilon < \epsilon'$. By monotonicity of measure, (i) $\Rightarrow$ (ii). By a similar argument, we can prove that (ii) $\Rightarrow$ (i) also holds.

STEP 2. Note that

$$m\left(\{x \in E \mid \sup_{k\geqq j}\{f_k(x)\} > \epsilon'\}\right) = m\left(\bigcup_{k\geqq j}^{\infty}\{x \in E \mid \{f_k(x)\} > \epsilon'\}\right).$$

So we have

$$\lim_{j\to\infty} m\left(\{x \in E \mid \sup_{k\geqq j}\{f_k(x)\} > \epsilon'\}\right) = \lim_{j\to\infty} m\left(\bigcup_{k\geqq j}^{\infty}\{x \in E \mid \{f_k(x)\} > \epsilon'\}\right) = 0.$$

By the Extra Theorem,

$$\lim_{j\to\infty} m\left(\bigcup_{k\geqq j}^{\infty}\{x\in E\mid \{f_k(x)\}>\epsilon'\}\right)=0 \Leftrightarrow f_k(x)\xrightarrow{\text{a.u}} f(x) \text{ on } E.$$

Let us recall that $f_k(x) \xrightarrow{\text{a.u}} f(x)$ always implies $f_k(x) \xrightarrow{\text{a.e}} f(x)$. By Egorov's theorem, when $m(E) < \infty$, $f_k(x) \xrightarrow{\text{a.e}} f(x)$ on E implies $f_k(x) \xrightarrow{\text{a.u}} f(x)$ on E. So if $m(E) < \infty$, $f_k(x) \xrightarrow{\text{a.u}} f(x) \Leftrightarrow f_k(x) \xrightarrow{\text{a.e}} f(x)$. Now the proof is complete.

□

71 (Exercise 7) Since $m([a,b]) = b-a < \infty$, by Egorov's theorem, there exists $E_n \in \mathscr{M}; m([a,b]\backslash E_n) < \frac{1}{n}$ s.t $f_k(x) \xrightarrow{u} f(x)$ on E_n. $m([a,b]\backslash\bigcup_{n=1}^{\infty} E_n) \leqq m([a,b]\backslash E_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$ so $m([a,b]\setminus\bigcup_{n=1}^{\infty} E_n) = 0$. Now the proof is complete. □

72 (Exercise 8) We may suppose $|f(x)|, |g(x)| < \infty$ a.e $x \in E$. By triangular inequality, we have

$$\begin{aligned} & |f_k + g_k - f - g| \geqq \epsilon \\ \Rightarrow\ & |f_k - f| + |g_k - g| \geqq \epsilon. \end{aligned}$$

And then $|f_k - f| \geqq \frac{\epsilon}{2}$ or $|g_k - g| \geqq \frac{\epsilon}{2}$. So we have

$$\begin{aligned} & m(\{x\in E\mid |f_k(x)+g_k(x)-f(x)-g(x)|\geqq\epsilon\}) \\ \leqq\ & m\left(\left\{x\in E\mid |f_k(x)-f(x)|\geqq\frac{\epsilon}{2}\right\}\cup\left\{x\in E\mid |g_k(x)-g(x)|\geqq\frac{\epsilon}{2}\right\}\right) \\ \overset{*1}{\leqq}\ & m\left(\left\{x\in E\mid |f_k(x)-f(x)|\geqq\frac{\epsilon}{2}\right\}\right)+m\left(\left\{x\in E\mid |g_k(x)-g(x)|\geqq\frac{\epsilon}{2}\right\}\right)\overset{*2}{\to} 0 \end{aligned}$$

- (∗1) By sub-additivity
- (∗2) $f_k(x) \xrightarrow{m} f(x)$, $g_k(x) \xrightarrow{m} g(x)$ on E.

□

73 (Exercise 9)

STEP 1. (⇒) Suppose that $f_k(x) \xrightarrow{m} f(x)$. Let $\epsilon > 0$ be an arbitrary positive number. Note that

$$\begin{aligned} & \limsup_{k\to\infty}\inf_{a>0}\{a+m(\{x\in E\mid |f_k(x)-f(x)|>a\})\} \\ \overset{*1}{\leqq}\ & \lim_{k\to\infty}(\epsilon+m(\{x\in E\mid |f_k(x)-f(x)|>\epsilon\})) \\ \overset{*2}{=}\ & \epsilon \end{aligned}$$

- (∗1) Since it takes $\inf_{a>0}(\cdots)$, the value is less than or equal to the case of $a=\epsilon$.
- (∗2) $f_k(x) \xrightarrow{m} f(x)$.

The left hand side is less than an arbitrary positive number $\epsilon > 0$. So we conclude that

$$\lim_{k\to\infty} \inf_{a>0}\{a + m(\{x \in E \mid |f_k(x) - f(x)| > a\})\} = 0.$$

STEP 2. $(\Leftarrow)$ Suppose that

$$\lim_{k\to\infty} \inf_{a>0}\{a + m(\{x \in E \mid |f_k(x) - f(x)| \geqq a\})\} = 0.$$

Let $\epsilon, \delta > 0$ be an arbitrary positive number. Let $\epsilon^* \stackrel{\text{def}}{=} \min\{\epsilon, \delta\}$. We have $K_{\epsilon,\delta}$ s.t $\forall k \geqq K$,

$$\inf_{a>0}\{a + m(\{x \in E \mid |f_k(x) - f(x)| \geqq a\})\} < \epsilon^*.$$

For each $k \geqq K$, we can find at least one a_k s.t

$$a_k + m(\{x \in E \mid |f_k(x) - f(x)| \geqq a_k\}) < \epsilon^*.$$

From this inequality, it is easy to find out that $a_k < \epsilon^*$ because $m(\cdots) \geqq 0$. Therefore $a_k < \epsilon$. So we have

$$\begin{aligned} & m(\{x \in E \mid |f_k(x) - f(x)| \geqq \epsilon\}) \\ \leqq\ & m(\{x \in E \mid |f_k(x) - f(x)| \geqq a_k\}) \\ <\ & a_k + m(\{x \in E \mid |f_k(x) - f(x)| \geqq a_k\}) < \epsilon^* \leqq \delta \end{aligned}$$

for all $k \geqq K$. This implies that $f_k(x) \stackrel{m}{\to} f(x)$.

□

74 (Exercise 10) In this question we want to show $f_n(x) \stackrel{m}{\to} f(x)$ on $[0,1] \Rightarrow \forall x_0 \in C(f), f_n(x_0) \to f(x_0)$. We show the contraposition. (If we want to prove $A \Rightarrow B$, we may also prove $\neg B \Rightarrow \neg A$)

In other words, we show that $\exists x_0 \in C(f)$ s.t $f_n(x_0) \not\to f(x_0) \Rightarrow f_n(x) \stackrel{m}{\not\to} f(x)$ on $[0,1]$. Note that $f_n(x_0) \to f(x_0)$ means that

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} \text{ s.t } \forall n \geqq N, |f_n(x_0) - f(x_0)| < \epsilon.$$

So $f_n(x_0) \not\to f(x_0)$ means that

$$\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists n \geqq N \text{ s.t } |f_n(x_0) - f(x_0)| \geqq \epsilon.$$

Hint: First, swap $\forall$ and $\exists$. Then take the negation of the final part of the statement.

STEP 1. Since $\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists n \geqq N$ s.t $|f_n(x_0) - f(x_0)| \geqq \epsilon$, we can find a subsequence $\{n_k\}_{k \geq 1}$ s.t $|f_{n_k}(x_0) - f(x_0)| \geqq \epsilon$ for all $k \geqq 1$. So $f_{n_k}(x_0) - f(x_0) \geqq \epsilon$ or $f_{n_k}(x_0) - f(x_0) \leqq -\epsilon$ for all $k \geqq 1$. At least one of the following statements holds.

- There exist infinitely many k s.t $f_{n_k}(x_0) - f(x_0) \geqq \epsilon$.
- There exist infinitely many k s.t $f_{n_k}(x_0) - f(x_0) \leqq -\epsilon$.

First suppose there exist infinitely many k s.t $f_{n_k}(x_0) - f(x_0) \geqq \epsilon$. (Even if we assume the second case, the proof is similar. So we only assume the first case.) So we can find a further subsequence n_{k_m} s.t $f_{n_{k_m}}(x_0) - f(x_0) \geqq \epsilon$.

STEP 2. Let us recall that $x_0 \in C(f)$. Therefore $\exists \delta > 0$ s.t $\forall x \in (x_0 - \delta, x_0 + \delta)$, $|f(x) - f(x_0)| < \frac{\epsilon}{2}$. So $\forall x \in (x_0, x_0 + \delta)$, $-\frac{\epsilon}{2} < f(x) - f(x_0) < \frac{\epsilon}{2}$. This implies that $f(x_0) + \frac{\epsilon}{2} > f(x)$. Moreover, we have $f(x_0) + \epsilon > f(x) + \frac{\epsilon}{2}$ by adding $\frac{\epsilon}{2}$ to the both sides.

STEP 3. Since for each n, $f_n(x)$ is a monotone increasing function. (i.e $x < x' \Rightarrow f(x) \leqq f(x')$.) So $\forall x \in (x_0, x_0 + \delta)$, $f_{n_{k_m}}(x) \geqq f_{n_{k_m}}(x_0)$. By Step1, $f_{n_{k_m}}(x) \geqq f_{n_{k_m}}(x_0) \geqq f(x_0) + \epsilon$. By Step2, $f(x_0) + \epsilon > f(x) + \frac{\epsilon}{2}$. Therefore, we have $x \in (x_0, x_0 + \delta) \Rightarrow f_{n_{k_m}}(x) - f(x) > \frac{\epsilon}{2} \Rightarrow |f_{n_{k_m}}(x) - f(x)| \geqq \frac{\epsilon}{2}$. So we have

$$m\left(\left\{x \in E \mid |f_{n_{k_m}}(x) - f(x)| \geqq \frac{\epsilon}{2}\right\}\right) \geqq \delta > 0$$

By taking lim inf,

$$\liminf_{m\to\infty} m\left(\left\{x \in E \mid |f_{n_{k_m}}(x) - f(x)| \geqq \frac{\epsilon}{2}\right\}\right) \geqq \delta > 0.$$

This implies that $f_{n_{k_m}}(x) \stackrel{\text{m}}{\nrightarrow} f(x)$. However, if $f_n(x) \xrightarrow{m} f(x)$, then for any subsequence $n'(k)$, $f_{n'(k)}(x) \xrightarrow{m} f(x)$. So from the discussion above, we conclude that $f_n(x) \stackrel{\text{m}}{\nrightarrow} f(x)$.

□

75 (Exercise 11) We can find $G_n \in \mathscr{O}^d$ s.t $m(G_n) < \frac{1}{n}$ and $f(x) \in C(\mathbb{R}^d \setminus G_n)$. Let $H \stackrel{\text{def}}{=} \bigcap_{n=1}^{\infty} G_n$. Then $\{x \in \mathbb{R}^d \setminus H \mid f(x) > t\} = \bigcup_{n=1}^{\infty}\{x \in \mathbb{R}^d \setminus G_n \mid f(x) > t\} \in \mathscr{M}$. This is because $f(x)$ is continuous on $\mathbb{R}^d \setminus G_n$ hence there exists an open set A_n s.t $\{x \in \mathbb{R}^d \setminus G_n \mid f(x) > t\} = (\mathbb{R}^d \setminus G_n) \cap A_n$.

Finally, $\{x \in \mathbb{R}^d \mid f(x) > t\} = \{x \in \mathbb{R}^d \setminus H \mid f(x) > t\} \cup \{x \in H \mid f(x) > t\} \in \mathscr{M}$ because H and its subset are measure zero sets. □

76 (Exercise 12) $\{x \in E \mid |f_k(x)g_k(x)| \geqq \epsilon\} \subset \{x \in E \mid |f_k(x)| \geqq \sqrt{\epsilon}\} \cup \{x \in E \mid |g_k(x)| \geqq \sqrt{\epsilon}\}$. By sub-additivity,

$$\begin{aligned} & \lim_{k\to\infty} m(\{x \in E \mid |f_k(x)g_k(x)| \geqq \epsilon\}) \\ \leqq\ & \lim_{k\to\infty} m(\{x \in E \mid |f_k(x)| \geqq \sqrt{\epsilon}\} \cup \{x \in E \mid |g_k(x)| \geqq \sqrt{\epsilon}\}) \\ \leqq\ & \lim_{k\to\infty} \left\{m(\{x \in E \mid |f_k(x)| \geqq \sqrt{\epsilon}\}) + m(\{x \in E \mid |g_k(x)| \geqq \sqrt{\epsilon}\})\right\} = 0. \end{aligned}$$

.

□

77 (Exercise 13) Let us recall that $f_k(x) \xrightarrow{m} f(x)$ if and only if $\forall k_l$ (a subsequence), $\exists k_{l_m}$ (a further subsequence) s.t $f_{k_{l_m}}(x) \xrightarrow{\text{a.u}} f(x)$.

STEP 1. Let k_l be an arbitrary subsequence. Since $f_k(x) \xrightarrow{m} f(x)$ on $[a,b]$, there exists k_{l_m} (a further subsequence) s.t $f_{k_{l_m}}(x) \xrightarrow{\text{a.u}} f(x)$ on $[a,b]$. Let us recall that $m([a,b]) < \infty$ hence $\xrightarrow{\text{a.u}}$ if and only if $\xrightarrow{\text{a.e}}$. So $f_{k_{l_m}}(x) \xrightarrow{\text{a.e}} f(x)$ on $[a,b]$.

STEP 2. By assumption $g(x)$ is continuous on $[a,b]$, $g \circ f_{k_{l_m}}(x) \xrightarrow{\text{a.e}} g \circ f(x)$ on $[a,b]$. Again, $\xrightarrow{\text{a.e}}$ if and only if $\xrightarrow{\text{a.u}}$ so $g \circ f_{k_{l_m}}(x) \xrightarrow{\text{a.u}} g \circ f(x)$ on $[a,b]$. Therefore we may say that $\forall k_l$ (a subsequence) $\exists k_{l_m}$ (a further subsequence) s.t $g \circ f_{k_{l_m}} \xrightarrow{\text{a.u}} g \circ f(x)$ on $[a,b]$. This implies $g \circ f_k(x) \xrightarrow{m} g \circ f(x)$ on $[a,b]$.

(Notice) In the future, we will provide a counterexample in the case of $[a,\infty)$.

□

78 (Exercise 14)

STEP 1. Let $\{F_n\}_{n \geqq 1}$ be a sequence of closed sets with $m(E \setminus F_n) < \frac{1}{n}; f(x) \in C(F_n)$. Then $m(E \setminus \bigcup_{n=1}^{\infty} F_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$ hence $m(E \setminus \bigcup_{n=1}^{\infty} F_n) = 0$.

STEP 2. $\{x \in \mathbb{R}^d \mid f(x) > t\} = \{x \in \mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} F_n \mid f(x) > t\} \cup \{x \in \bigcup_{n=1}^{\infty} F_n \mid f(x) > t\}$. $= \{x \in \mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} F_n \mid f(x) > t\} \cup \bigcup_{n=1}^{\infty}\{x \in F_n \mid f(x) > t\}$. Since $\{x \in \mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} F_n \mid f(x) > t\} \subset \mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} F_n$, so this is a measure zero set. Moreover since $f(x) \in C(F_n)$ for each $n \in \mathbb{N}$, we have $\{x \in F_n \mid f(x) > t\} \in \mathscr{M}$ and hence $\bigcup_{n=1}^{\infty}\{x \in F_n \mid f(x) > t\} \in \mathscr{M}$ Now the proof is complete.

□

79 (Exercise 15)

STEP 1. In this question, we do not know if $f(x)$ is a measurable function. We define convergence in measure to a sequence of measurable functions $\{f_k(x)\}_{k \geqq 1}$ and a measurable function $f(x)$. Therefore we should not say $f_n(x) \xrightarrow{m} f(x)$ from the assumption.

Let us look back on the equivalent statement on convergence a.e. Let $\{f_k(x)\}_{k \geqq 1}$ be a sequence of measurable functions. (We do not suppose measurability of $f(x)$.) Then $f_k(x) \xrightarrow{\text{a.e}} f(x)$ on E if and only if for all $\epsilon > 0$, we have

$$m\left(\bigcap_{m=1}^{\infty} \bigcup_{k \geqq m}^{\infty} \{x \in E \mid |f_k(x) - f(x)| \geqq \epsilon\}\right) = 0.$$

STEP 2. We pick a subsequence $\{n_k\}_{k \geqq 1}$ s.t

$$m^*(\{x \in [a,b] \mid |f_{n_k}(x) - f(x)| > \epsilon\}) \leqq \frac{1}{2^{k+1}}.$$

We still do not know if the set is measurable or not so use $m^*(\cdot)$. Though we do not know measurability, an outer measure m^* has sub-additivity. So

$$m^*\left(\bigcup_{k \geqq m}^{\infty} \{x \in [a,b] \mid |f_{n_k}(x) - f(x)| \geqq \epsilon\}\right) \leqq \frac{1}{2^m}.$$

Moreover,

$$m^*\left(\bigcap_{m=1}^{\infty} \bigcup_{k \geqq m}^{\infty} \{x \in [a,b] \mid |f_{n_k}(x) - f(x)| \geqq \epsilon\}\right) \leqq \frac{1}{2^m},\ \forall m \in \mathbb{N}.$$

This implies that

$$m^*\left(\bigcap_{m=1}^{\infty}\bigcup_{k\geqq m}^{\infty}\{x\in[a,b]\mid|f_{n_k}(x)-f(x)|\geqq\epsilon\}\right)=0.$$

A measure zero set is measurable, so $\bigcap_{m=1}^{\infty}\bigcup_{k\geqq m}^{\infty}\{x\in[a,b]\mid|f_{n_k}(x)-f(x)|\geqq\epsilon\}\in\mathscr{M}$. Let us recall the extra theorem. $f_{n_k}(x)\xrightarrow{\text{a.e}}f(x)$ on $[a,b]$ if and only if

$$m\left(\bigcap_{m=1}^{\infty}\bigcup_{k\geqq m}^{\infty}\{x\in[a,b]\mid|f_{n_k}(x)-f(x)|\geqq\epsilon\}\right)=0,\ \forall\epsilon>0$$

Since $f_{n_k}(x)\xrightarrow{\text{a.e}}f(x)$, so $f(x)$ is measurable.

□

80 (Exercise 16) See the extra theorem. □

CHAPTER 4

Solutions

§ 4.1

1 (Definition 4.1)

$$\int_E f(x)dx \overset{\text{def}}{=} \sum_{i=1}^{p} c_i m(E \cap A_i).$$

$\square$

2 (Theorem 4.1)

(1) By definition,

$$\int_E (cf(x))dx = \sum_{i=1}^{p} (ca_i)m(E \cap A_i) = c\sum_{i=1}^{p} a_i m(E \cap A_i) = c\int_E f(x)dx.$$

(2) Since $\mathbb{R}^d = \bigcup_{i=1}^{p} A_i = \bigcup_{j=1}^{q} B_j$,

$$f(x) + g(x) = \sum_{i=1}^{p}\sum_{j=1}^{q} (a_i + b_j)\chi_{A_i \cap B_j}(x).$$

This is also a non-negative Lebesgue measurable simple function. By definition,

$$\int_E (f(x) + g(x))dx = \sum_{i=1}^{p}\sum_{j=1}^{q} (a_i + b_j)m(E \cap A_i \cap B_j).$$

Again, since $\mathbb{R}^d = \bigcup_{i=1}^{p} A_i = \bigcup_{j=1}^{q} B_j$,

$$\sum_{i=1}^{p}\sum_{j=1}^{q} a_i m(E \cap A_i \cap B_j) \overset{(*1)}{=} \sum_{i=1}^{p} a_i m(E \cap A_i) = \int_E f(x)dx.$$

- $(*1)$ is because $\{E \cap A_i \cap B_j\}_{j=1}^q$ are disjoint with each other, and $\bigcup_{j=1}^q E \cap A_i \cap B_j = E \cap A_i$ because $\bigcup_{j=1}^q B_j = \mathbb{R}^d$.

Simlarly,

$$\sum_{i=1}^p \sum_{j=1}^q b_j m(E \cap A_i \cap B_j) = \sum_{j=1}^q b_j m(E \cap B_j) = \int_E g(x)dx,$$

so the right hand side becomes $\int_E f(x)dx + \int_E g(x)dx$.

(3)

$$\begin{aligned}
\int_E f(x)dx = \sum_{i=1}^p a_i m(E \cap A_i) &= \sum_{i=1}^p \sum_{j=1}^q a_i m(E \cap A_i \cap B_j) \\
&\overset{(*2)}{\leqq} \sum_{i=1}^p \sum_{j=1}^q b_j m(E \cap A_i \cap B_j) \\
&= \sum_{j=1}^q \sum_{i=1}^p b_j m(E \cap A_i \cap B_j) \\
&= \sum_{j=1}^q b_j m(E \cap B_j) = \int_E g(x)dx
\end{aligned}$$

- $(*2)$ If $x \in A_i \cap B_j \neq \emptyset$, then $f(x) \leqq g(x)$, hence $a_i \leqq b_j$. For a given pair of A_i, B_j with $A_i \cap B_j \neq \emptyset$, $a_i \leqq b_j$. Therefore $a_i \cdot m(E \cap A_i \cap B_j) \leqq b_j \cdot m(E \cap A_i \cap B_j)$ if $A_i \cap B_j \neq \emptyset$. And since $m(\emptyset) = 0$, when $A_i \cap B_j = \emptyset$, the equality still holds.

□

3 (Theorem 4.2) Let $f(x) \overset{\text{def}}{=} \sum_{i=1}^p c_i \chi_{A_i}(x)$ where $\mathbb{R}^d = \bigcup_{i=1}^p A_i$ and $A_i \in \mathscr{M}$, $A_i \cap A_j = \emptyset$ if $i \neq j$.

$$\begin{aligned}
\lim_{k\to\infty} \sum_{i=1}^p c_i m(E_k \cap A_i) &= \sum_{i=1}^p \lim_{k\to\infty} c_i m(E_k \cap A_i) \\
&\overset{(*)}{=} \sum_{i=1}^p c_i m(E \cap A_i) = \int_E f(x)dx
\end{aligned}$$

.

- $(*)$ $E_k \cap A_i \nearrow E \cap A_i$ as $k \to \infty$, so $m(E_k \cap A_i) \nearrow m(E \cap A_i)$ as $k \to \infty$. (See Theorem 2.7)

□

4 (Definition 4.2) Let $\mathscr{G}$ be a collection of non-negative Lebesgue measurable simple functions defined on $\mathbb{R}^d$. (If we regard E as a universal set, we may also consider that simple functions in $\mathscr{G}$ are defined on E.) Let $f(x)$ be a non-negative measurable function defined on $E \in \mathscr{M}$.

$$\mathscr{G}_f \stackrel{\text{def}}{=} \{g \in \mathscr{G} \mid g(x) \leqq f(x),\ \forall x \in E\}.$$

Then we define

$$\int_E f(x)dx \stackrel{\text{def}}{=} \sup_{g \in \mathscr{G}_f} \left\{ \int_E g(x)dx \right\}.$$

If $\int_E f(x)dx < \infty$, we say that f is integrable on E. □

5 (Extra Theorem) Let $f(x)$ be a non-negative Lebesgue measurable simple function defined on $\mathbb{R}^d$. Let $S_1 \stackrel{\text{def}}{=} \int_E f(x)dx$ by Definition 4.1, and let $S_2 \stackrel{\text{def}}{=} \int_E f(x)dx$ by Definition 4.2.

STEP 1. ($S_1 \leqq S_2$) Since $f(x) \in \mathscr{G}_f$, $S_1 \in \{\int_E g(x)dx \mid g \in \mathscr{G}_f\}$, hence $S_1 \leqq S_2 \stackrel{\text{def}}{=} \sup\{\int_E g(x)dx \mid g \in \mathscr{G}_f\}$.

STEP 2. ($S_1 \geqq S_2$) Let us recall Theorem 4.1. (In definition 4.1, if $f \leqq g$, then $\int f \leqq \int g$.) Since $\forall g(x) \in \mathscr{G}_f$, $g(x) \leqq f(x)$ holds. And $\forall S \in \{\int_E g(x)dx \mid g \in \mathscr{G}_f\}$, we have $S \leqq S_1$. By taking sup of the left hand side, we have $S_2 \leqq S_1$.

□

6 (Some Properties derived from Definition 4.2) Let $\mathscr{G}$ be a collection of non-negative measurable simple function defined on $\mathbb{R}^d$ (or E).

(1) Let $\mathscr{G}_1 \stackrel{\text{def}}{=} \{h \in \mathscr{G} \mid h(x) \leqq f(x), \forall x \in E\}$. Let $\mathscr{G}_2 \stackrel{\text{def}}{=} \{h \in \mathscr{G} \mid h(x) \leqq g(x), \forall x \in E\}$. Since $f(x) \leqq g(x)$, $\mathscr{G}_1 \subset \mathscr{G}_2$. So $\left\{\int_E h(x)dx\right\}_{h \in \mathscr{G}_1} \subset \left\{\int_E h(x)dx\right\}_{h \in \mathscr{G}_2}$ Therefore

$$\int_E f(x)dx = \sup_{h \in \mathscr{G}_1} \left\{ \int_E h(x)dx \right\} \leqq \sup_{h \in \mathscr{G}_2} \left\{ \int_E h(x)dx \right\} = \int_E g(x)dx$$

(2) By the previous result, we have $\int_E f(x)dx \leqq \int_E g(x) < \infty$. So $f(x)$ is also integrable.

(3) Let $\mathscr{G}_1 \stackrel{\text{def}}{=} \{h_1 \in \mathscr{G} \mid h_1(x) \leqq f(x), \forall x \in A\}$. Let $\mathscr{G}_2 \stackrel{\text{def}}{=} \{h_2 \in \mathscr{G} \mid h_2(x) \leqq f(x)\chi_A(x), \forall x \in E\}$. By definition,

$$\begin{aligned} S_1 &\stackrel{\text{def}}{=} \int_A f(x)dx = \sup_{h_1 \in \mathscr{G}_1} \left\{ \int_A h_1(x)dx \right\} \\ S_2 &\stackrel{\text{def}}{=} \int_E f(x)\chi_A(x)dx = \sup_{h_2 \in \mathscr{G}_2} \left\{ \int_E h_2(x)dx \right\}. \end{aligned}$$

STEP 1. ($S_1 \leqq S_2$) We pick a function $h_1(x) \in \mathscr{G}_1$ arbitrarily and suppose that $h_1(x) = \sum_{i=1}^{p} a_i \chi_{A_i}(x)$ where $\mathbb{R}^d = \bigcup_{i=1}^{p} A_i$, where $\{A_i\}_{i=1}^{p} \subset \mathscr{M}$ are mutually disjoint. By assumption, $h_1(x) \leqq f(x), \forall x \in A$. So $h_1(x) \cdot \chi_A(x) \leqq f(x) \cdot \chi_A(x), \forall x \in E$. This implies that $h_1(x) \cdot \chi_A(x) \in \mathscr{G}_2$.

Since

$$\int_A h_1(x) = \sum_{i=1}^{p} a_i m(A \cap A_i),$$

and

$$\begin{aligned}\int_E h_1(x) \cdot \chi_A(x)dx &\overset{*1}{=} \sum_{i=1}^{p} a_i m(E \cap A \cap A_i) \\ &\overset{*2}{=} \sum_{i=1}^{p} a_i m(A \cap A_i),\end{aligned}$$

- $(*1)$ $h_1(x) \cdot \chi_A(x) = \sum_{i=1}^p a_i \chi_{A\cap A_i}(x) = \sum_{i=1}^p a_i \chi_{A\cap A_i}(x) + 0 \cdot \chi_{\mathbb{R}^d \setminus A}$ is also a measurable simple function.
- $(*2)$ $A \subset E$.

hence

$$\int_A h_1(x)dx = \int_E h_1(x) \cdot \chi_A(x)dx \overset{*3}{\leqq} S_2.$$

- $(*3)$ Let $h_2(x) \overset{\text{def}}{=} h_1(x) \cdot \chi_A(x) \in \mathscr{G}_2$. So $\int_E h_2(x)dx \leqq \sup_{h\in\mathscr{G}_2}\{\int_E h(x)dx\}$.

By taking sup with respect to $h_1(x)$ on the left hand side in the above inequality, we have $S_1 \leqq S_2$.

STEP 2. $(S_1 \geqq S_2)$ We pick a function $h_2(x) \in \mathscr{G}_2$ arbitrarily, and suppose that $h_2(x) = \sum_{i=1}^p a_i \chi_{A_i}(x)$ where $\mathbb{R}^d = \bigcup_{i=1}^p A_i$. ($\{A_i\}_{i=1}^p \subset \mathscr{M}$ are mutually disjoint.) By assumption, $h_2(x) \leqq f(x)\chi_A(x)$. This implies that if $x \notin A$, $h_2(x) = 0$. So it follows that $h_2(x) = h_2(x) \cdot \chi_A(x) = \sum_{i=1}^p a_i \chi_{A\cap A_i}(x)$, and $h_2(x) \leqq f(x)$ for $x \in A$, so $h_2(x) \in \mathscr{G}_1$ if we regard $h_2(x)$ as a function defined on A.

$$\begin{aligned}\int_E h_2(x)dx &= \int_E h_2(x) \cdot \chi_A(x) \\ &= \sum_{i=1}^{p} a_i m(E \cap A \cap A_i) \\ &= \sum_{i=1}^{p} a_i m(A \cap A_i) \\ &= \int_A h_2(x)dx \overset{*4}{\leqq} S_1.\end{aligned}$$

- $(*4)$ $h_2(x) \in \mathscr{G}_1$, so $\int_A h_2(x)dx \leqq \sup_{h\in\mathscr{G}_1}\{\int_A h(x)dx\} = S_1$.

Finally, by taking sup with respect to h_2 on the left hand side, we have $S_2 \leqq S_1$.

(4)

STEP 1. $(\Rightarrow)$ We pick an arbitrary measurable simple function $h(x) = \sum_{i=1}^p a_i \chi_{A_i}(x)$ s.t $h(x) \leqq f(x)$ on E. Since $f(x) = 0$ a.e $x \in E$, if $m(E \cap A_i) > 0$ then $a_i = 0$. This implies that either a_i or $m(E \cap A_i)$ is 0. Therefore $\int_E h(x)dx = \sum_{i=1}^p a_i m(E \cap A_i) = 0$. Even if we take $\sup_{h\in\mathscr{G}}\{\int_E h(x)dx\}$, it should still be 0.

STEP 2. ($\Leftarrow$) Since

$$\begin{aligned} 0 = \int_E f(x)dx &\geqq \int_E f(x)\cdot \chi_{\{x\in E | f(x) \geqq \frac{1}{n}\}}(x)dx \\ &\geqq \int_E \frac{1}{n}\chi_{\{x\in E | f(x) \geqq \frac{1}{n}\}}(x)dx \\ &= \frac{1}{n}m(\{x \in E \mid f(x) \geqq \frac{1}{n}\}), \end{aligned}$$

we have

$$m\left(\{x \in E \mid f(x) \geqq \frac{1}{n}\}\right) = 0,\ \forall n \in \mathbb{N}.$$

Therefore, by sub-additivty

$$\begin{aligned} m\left(\{x \in E \mid f(x) > 0\}\right) &= m\left(\bigcup_{n=1}^{\infty}\{x \in E \mid f(x)\frac{1}{n}\}\right) \\ &\leqq \sum_{n=1}^{\infty} m\left(\{x \in E \mid f(x) \geqq \frac{1}{n}\}\right) = 0. \end{aligned}$$

(5) Since E is a measure zero set, we may say that $f(x) = 0$ a.e $x \in E$. ($\because \{x \in E \mid f(x) > 0\} \subset E$) So $\int_E f(x)dx = 0$.

□

7 (Theorem 4.3) Let $E_k \overset{\text{def}}{=} \{x \in E \mid f(x) > k\}$. Then

$$E_k \searrow \bigcap_{k=1}^{\infty} E_k = \{x \in E \mid f(x) = \infty\}.$$

Next,

$$km(E_k) = \int_{E_k} k\chi_E dx \leqq \int_{E_k} f(x)dx \leqq \int_E f(x)dx < \infty,$$

hence

$$m(E_k) \leqq \frac{1}{k}\int_E f(x)dx < \infty.$$

This implies that $m(E_1) < \infty$ and $\lim_{k\to\infty} m(E_k) = 0$. Therefore,

$$\lim_{k\to\infty} m(E_k) = m\left(\bigcap_{k=1}^{\infty} E_k\right) = m\left(\{x \in E \mid f(x) = \infty\}\right) = 0.$$

So we have the desired conclusion. □

8 (Theorem 4.4) Let $\mathscr{G}$ be a collection of non-neagative Lebesgue measurable simple functions define on $E \in \mathscr{M}$. And let $\mathscr{G}_f \overset{\text{def}}{=} \{g \in \mathscr{G} \mid g \leqq f\}$.

STEP 1. ($\leqq$) First $\int_E f_k(x)dx$ is increasing so the limit exists. Since $\int_E f_k(x)dx \leqq \int_E f(x)dx$ for all $k = 1, 2\cdots$, so $\lim_{k\to\infty} \int_E f_k(x)dx$.

STEP 2. ($\geqq$) Let us recall that

$$\int_E f(x)dx \stackrel{\text{def}}{=} \sup_{g\in\mathscr{G}_f} \left\{ \int_E g(x)dx \right\}.$$

So it is enough for us to show that $\forall g \in \mathscr{G}_f$,

$$\lim_{k\to\infty} \int_E f_k(x)dx \geqq \int_E g(x)dx.$$

Let $\alpha \in (0,1)$ and we define

$$E_k^{(\alpha)} \stackrel{\text{def}}{=} \{x \in E \mid f_k(x) > \alpha g(x)\}.$$

Since $f_k \nearrow f$ and $g \leqq f$ so we have

$$E_k^{(\alpha)} \nearrow \{x \in E \mid f(x) > \alpha g(x)\} = E.$$

Finally,

$$\begin{aligned}
\lim_{k\to\infty} \int_E f_k(x)dx &\stackrel{(*1)}{\geqq} \lim_{k\to\infty} \int_{E_k^{(\alpha)}} f_k(x)dx \\
&\stackrel{(*2)}{\geqq} \lim_{k\to\infty} \int_{E_k^{(\alpha)}} \alpha g(x)dx \\
&\stackrel{(*3)}{=} \int_E \alpha g(x)dx \\
&\stackrel{(*4)}{=} \alpha \int_E g(x)dx
\end{aligned}$$

By taking $\alpha \nearrow 1$, we have the desired result. Hint.

- $(*1)$ Let $A, B \in \mathscr{M}, A \subset B$, and let $f(x)$ be a non-negative Lebesuge measurable function. Then $\int_A f(x)dx = \int_B f(x) \cdot \chi_A(x) \leqq \int_B f(x)dx$.
- $(*2)$ When $x \in E_k^{(\alpha)}$, $f_k(x) > \alpha g(x)$
- $(*3)$ $\lim_{k\to\infty} \int_{E_k^{(\alpha)}} \alpha g dx = \int_E \alpha g dx$. This follows by Theorem 4.2.
- $(*4)$ Theorem 4.1.

□

9 (Theorem 4.5) According to the theorem in Chpter 3, we can find a sequence of non-negative measurable simple functions s.t $f_k(x) \nearrow f(x)$ and $g_k(x) \nearrow g(x)$. So $\{\alpha f_k(x) + \beta g_k(x)\}_{k\geqq 1}$ is also an increasing sequence of non-negatie measurable simple functions s.t $\alpha f_k(x) + \beta g_k(x) \nearrow \alpha f(x) + \beta g(x)$.

By monotone convergence theorem (Theorem 4.4),

$$\lim_{k\to\infty}\int_E(\alpha f_k(x)+\beta g_k(x))dx=\int_E(\alpha f(x)+\beta g(x))dx.$$

By Theorem 4.1, when the functions are measurable simple functions, integral has linearity. So the left hand side is

$$\lim_{k\to\infty}\int_E(\alpha f_k(x)+\beta g_k(x))dx=\lim_{k\to\infty}\left(\alpha\int_E f_k(x)dx+\beta\int_E g_k(x)dx\right)$$

Again by monotone convergence theorem (Theorem 4.4), the right hand side is

$$\lim_{k\to\infty}\left(\alpha\int_E f_k(x)dx+\beta\int_E g_k(x)dx\right)=\alpha\int_E f(x)dx+\beta\int_E g(x)dx.$$

Now the proof is complete. □

10 (Example 2)

STEP 1. Let us pay attention to the fact that $f(x)\geqq 0$ for all $x\in E$ because $f_k(x)\geqq 0$ and $f_k(x)\to f(x)$ on E. Let $g_k(x)=f_1(x)-f_k(x)$. Then $g_k(x)\geqq 0$ so $g_k(x)$ is non-negative so $\{g_k(x)\}_{k\geqq 1}$ is an increasing sequence of non-negative measurable functions. $g_k(x)\to f_1(x)-f(x)$ on E. ($f_1(x)-f(x)\geqq 0$ for all $x\in E$.) By Theorem 4.4 (monotone convergence theorem), we have

$$\lim_{k\to\infty}\int_E g_k(x)dx=\int_E g(x)dx=\int_E(f_1(x)-f(x))dx.\ \cdots\ (*1)$$

STEP 2. We still can not say that $\int_E(f_1(x)-f(x))dx=\int_E f_1(x)dx-\int_E f(x)dx$ because Theorem 4.5 assumes that $\alpha,\beta>0$. However, according to linearity of integral with regard to non-negative measurable funtions, we have

$$\int_E((f_1(x)-f(x))+f(x))dx=\int_E(f_1(x)-f(x))dx+\int_E f(x)dx.$$

Since $\int_E f(x)dx\leqq\int_E f_k(x)dx<\infty$ (finite), we may subtract $\int_E f(x)dx$ from the both sides. So we have

$$\int_E((f_1(x)-f(x))+f(x))dx-\int_E f(x)dx=\int_E(f_1(x)-f(x))dx.$$

Therefore we have

$$\int_E f_1(x)dx-\int_E f(x)dx=\int_E(f_1(x)-f(x))dx.\ \cdots\ (*2)$$

Similarly we have

$$\int_E g_k(x)dx=\int_E f_1(x)dx-\int_E f_k(x)dx.\ \cdots\ (*3)$$

STEP 3.

$$\begin{aligned}\lim_{k\to\infty}\int_E g_k(x)dx &\overset{(*1)}{=} \int_E (f_1(x)-f(x))dx\\ &\overset{(*2)}{=} \int_E f_1(x)dx - \int_E f(x)dx,\\ \lim_{k\to\infty}\int_E g_k(x)dx &\overset{(*3)}{=} \lim_{k\to\infty}\left(\int_E f_1(x)dx - \int_E f_k(x)dx\right)\\ &= \int_E f_1(x)dx - \lim_{k\to\infty}\int_E f_k(x)dx.\end{aligned}$$

Since $\int_E f_1(x) < \infty$, we may subtract $\int_E f_1(x)dx$ from the both sides. By multiplying -1 to the both sides, we have the desired result.

□

11 (Example 3) Let $N \overset{\text{def}}{=} \{x \in E \mid f(x) \neq g(x)\} \in \mathscr{M}$. $m(N) = 0$.

$$\begin{aligned}\int_E f(x)dx &= \int_E (f(x)\chi_N(x) + f(x)\chi_{E\setminus N}(x))dx\\ &\overset{(*1)}{=} \int_E f(x)\chi_N(x)dx + \int_E f(x)\chi_{E\setminus N}(x))dx\\ &\overset{(*2)}{=} \int_E f(x)\chi_N(x)dx + \int_E g(x)\chi_{E\setminus N}(x))dx\\ &\overset{(*3)}{=} \int_N f(x)dx + \int_E g(x)\chi_{E\setminus N}(x))dx\\ &\overset{(*4)}{=} \quad 0 \quad + \int_E g(x)\chi_{E\setminus N}(x))dx\\ &\overset{(*5)}{=} \int_N g(x)dx + \int_E g(x)\chi_{E\setminus N}(x))dx\\ &\overset{(*6)}{=} \int_E g(x)\chi_N(x)dx + \int_E g(x)\chi_{E\setminus N}(x))dx\\ &\overset{(*7)}{=} \int_E (g(x)\chi_N(x)dx + g(x)\chi_{E\setminus N}(x))dx\\ &= \int_E g(x)dx.\end{aligned}$$

- $(*1), (*7)$ holds by Theorem 4.5.
- $(*2)$ $f(x) = g(x)$ on $E \setminus N$.
- $(*3), (*4), (*5), (*6)$ See the properties of integral derived from Definition 4.2. $\int_A h(x)dx = \int_E h(x)\chi_A(x)dx$ and $\int_A h(x)dx = 0$ if $m(A) = 0$ where $A \subset E, A \in \mathscr{M}$ and $h(x)$ is a non-negative Lebesgue measurable function.

□

12 (Supplement to Theorem 4.5 and Example 2) Let $\tilde{f}(x) \stackrel{\text{def}}{=} \lim_{k\to\infty} f(x)$. Then $\tilde{f}(x) = f(x)$ a.e $x \in E$. By Example 3, we have

$$\int_E \tilde{f}(x)dx = \int_E f(x)dx.$$

By Theorem 4.5 or Example 2, we also have

$$\int_E \tilde{f}(x)dx = \lim_{k\to\infty} \int_E f_k(x)dx.$$

Now the proof is complete. □

13 (Exercise 1)

(1) Since $(f_1^2 + f_2^2 + \cdots + f_m^2) \leqq (f_1 + f_2 \cdots + f_m)^2$ when $f_1, f_2 \cdots f_m \geqq 0$, we have

$$F(x) \leqq f_1(x) + f_2(x) \cdots + f_m(x).$$

The right hand side is integrable because $f_i(x)$ is integrable for each $i = 1, 2 \cdots m$. Therefore $F(x)$ is also integrable. (See the properties of integral derived from Definition 4.2.)

(2) It is enough for us to show that for every combination of (i, k), $(f_i(x)f_k(x))^{1/2}$ is integrable.

case 1. $(i = k)$ When $i = k$, then $(f_i(x)f_k(x))^{1/2} = f_i(x)$ so it is integrable.

case 2. $(i \neq k)$ When $i \neq k$, then $(f_i(x)f_k(x))^{1/2} \leqq \sqrt{2}(f_i(x)f_k(x))^{1/2} \leqq f_i(x) + f_k(x)$. (Take the square of the both sides and you will find that the right hand side is equal or greater than the left side.) The right hand side is integrable.

Now the proof is complete.

□

14 (Exercise 2) According to the properties of integral derived from the Definition 4.2, the right hand side is

$$\lim_{k\to\infty} \int_E f(x)\chi_{E_k}(x).$$

Let $g_k(x) = f(x)\chi_{E_k}(x)$. Then $\{g_k(x)\}_{k\geqq 1}$ is an increasing sequence of non-negative measurable functions and $g_k(x) \nearrow f(x)$. By monotone convergence theorem (Theorem 4.5), we have

$$\lim_{k\to\infty} \int_E f(x)\chi_{E_k}(x) = \lim_{k\to\infty} \int_E g_k(x) = \int_E f(x).$$

So the proof is complete.

□

15 (Exercise 3) By the hint we have

$$\limsup_{k\to\infty} \int_E (1 - \exp(-f_k(x)))dx \leqq \lim_{k\to\infty} \int_E f_k(x)dx = 0.$$

So the proof is complete. □

16 (Exercise 4) Let $g_n(x) \stackrel{\text{def}}{=} f(x)\chi_{\{x\in E|f(x)>n\}}(x)$. Then $\{g_n(x)\}_{n\geqq 1}$ is a decreasing sequence of non-negative measurable functions. Moreover $g_n(x)$ is also integrable because $f(x)$ is integrable and $0 \leqq g_n(x) \leqq f(x)$. $g_n(x) \searrow g(x) \stackrel{\text{def}}{=} \infty \cdot \chi_{\{x\in E|f(x)=\infty\}}(x)$. Since $f(x)$ is integrable, by Theorem 4.3, $f(x) < \infty$ a.e $x \in E$. Hence $m(\{x \in E \mid f(x) = \infty\}) = 0$. So we may say that $g(x) = 0$ a.e $x \in E$. By Example2 (and Example 3 in the second equality), we have

$$\lim_{k\to\infty} \int_E g_k(x)dx = \int_E g(x)dx = \int_E 0dx = 0.$$

This implies that $\forall\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ s.t

$$\int_E g_N(x)dx < \epsilon$$

So the proof is complete.

□

17 (Exercise 5) We use monotone convergence theorem (Theorem 4.4).

STEP 1. We show that $a_n^{(x)} \stackrel{\text{def}}{=} \left(1 + \frac{x}{n}\right)^n$ is increasing with respect to n for all $x \geqq 0$. (i.e $a_n^{(x)} \leqq a_{n+1}^{(x)}$). Let $g_x(t) \stackrel{\text{def}}{=} \ln\left(1 + \frac{x}{t}\right)^x$, $(t > 0)$.. Then

$$\begin{aligned} g_x'(t) &= \ln\left(1 + \frac{x}{t}\right) - \frac{\frac{x}{t}}{1 + \frac{x}{t}} \\ g_x''(t) &= -\frac{t}{(t+x)^2} < 0 \end{aligned}$$

$g_x'(t)$ is monotone decreasing in $t \in (0, \infty)$ and $\lim_{t\to\infty} g_x'(t) = 0$. This implies that$g_x'(t) > 0$. Therefore $g_x(t)$ is monotone increasing. So $a_n^{(n)}$ is also monotone increasing with respect to n for all $x \geqq 0$.

STEP 2. By monotone convergence theorem (Theorem 4.4), we have

$$\begin{aligned} &\lim_{n\to\infty} \int_{[0,n]} \left(1 + \frac{x}{n}\right)^n \exp(-2x)dx \\ \stackrel{*1}{=}& \lim_{n\to\infty} \int_{[0,\infty)} \left(1 + \frac{x}{n}\right)^n \chi_{[0,n]}(x)\exp(-2x)dx \\ \stackrel{*2}{=}& \int_{[0,\infty)} \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n \chi_{[0,n]}(x)\exp(-2x)dx \\ =& \int_{[0,\infty)} \exp(x)\chi_{[0,\infty)}(x)\exp(-2x)dx \\ =& \int_{[0,\infty)} \exp(-x)dx \end{aligned}$$

- $(*1)$ We may consider that $\left(1 + \frac{x}{n}\right)^n \chi_{[0,n]}(x)\exp(-2x)$ is a non-negative Lebesgue measurable function defined on $E = [0, \infty)$.

- (∗2) We apply monotone convergence theorem here.

□

18 (Exercise 6)

$$f_n(x) \overset{\text{def}}{=} x^n \to \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$

Since $m(\{1\}) = 0$, $f_n(x) \overset{\text{a.e}}{\longrightarrow} 0$ on $[0,1]$. Moreover, $0 \leqq f_{n+1}(x) \leqq f_n(x)$ on $[0,1]$. By Example2, we have

$$\lim_{n\to\infty} \int_{[0,1]} f_n(x)dx = \int_{[0,1]} 0dx = 0.$$

□

19 (Theorem 4.6) Let $S_k(x) \overset{\text{def}}{=} \sum_{i=1}^{k} f_i(x)$. Since $f_i(x)$ is non-negative measurable functions. Hence $S_k(x)$ is also non-negative measurable functions and $S_k(x) \leqq S_{k+1}(x), S_k(x) \nearrow \sum_{k=1}^{\infty} S_k(x)$ holds. By monotone convergence theorem (Theorem 4.4) we have

$$\lim_{k\to\infty} \int_E S_k(x)dx = \int_E \sum_{k=1}^{\infty} S_k(x)dx.$$

By Theorem 4.5 (integral has linearity), so the left hand side is

$$\lim_{k\to\infty} \int_E S_k(x)dx = \lim_{k\to\infty} \sum_{i=1}^{k} \int_E f_i(x)dx = \sum_{k=1}^{\infty} \int_E f_k(x)dx$$

□

20 (Corollary 4.7)

STEP 1. It is easy to verify that $\chi_E(x) = \sum_{k=1}^{\infty} \chi_{E_k}(x)$. First, suppose that $\chi_E(x) = 1$. Then $x \in E$ so $\exists k_0$ s.t $x \in E_{k_0}$. And $\{E_k\}_{k=1}^{\infty}$ are mutually disjoint, $\sum_{k=1}^{n} \chi_{E_k}(x) = 1$ for sufficiently large n. So $\sum_{k=1}^{\infty} \chi_{E_k}(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \chi_{E_k}(x) = 1$.

Second, suppose that $\sum_{k=1}^{\infty} \chi_{E_k}(x) = 1$. By the similar argument, we have $\chi_E(x) = 1$. Since the both sides only take 0 or 1, the argument above explains that $\chi_E(x) = \sum_{k=1}^{\infty} \chi_{E_k}(x)$.

STEP 2. Since $x \in E$, we have

$$\begin{aligned} f(x) &= f(x)\chi_E(x) \\ &\overset{*1}{=} f(x)\sum_{k=1}^{\infty}\chi_{E_k}(x) \\ &\overset{*2}{=} f(x)\lim_{n\to\infty}\sum_{k=1}^{n}\chi_{E_k}(x) \\ &\overset{*3}{=} \lim_{n\to\infty}\sum_{k=1}^{n} f(x)\chi_{E_k}(x) \\ &\overset{*4}{=} \sum_{k=1}^{\infty} f(x)\chi_{E_k}(x). \end{aligned}$$

- $(*1)$ By Step1.
- $(*2), (*4)$ By the definition of limit of summation.
- $(*3)$ if $\{a_n\}$ converges, then $\alpha \cdot \lim_{n\to\infty} a_n = \lim_{n\to\infty} \alpha \cdot a_n$.

Therefore,

$$\int_E f(x)dx = \int_E \sum_{k=1}^{\infty} f(x)\chi_{E_k}(x)dx$$

By Theorem 4.6 we have,

$$\int_E \sum_{k=1}^{\infty} f(x)\chi_{E_k}(x)dx = \sum_{k=1}^{\infty}\int_E f(x)\chi_{E_k}(x)dx.$$

Finally, by the properties of integral derived from Definition 4.2,

$$\sum_{k=1}^{\infty}\int_E f(x)\chi_{E_k}(x)dx = \sum_{k=1}^{\infty}\int_{E_k} f(x)dx.$$

□

21 (Example 4) By assumption, $\sum_{i=1}^{n}\chi_{E_i}(x) \geqq k$ for all $x \in [0,1]$. So

$$\int_{[0,1]}\sum_{i=1}^{n}\chi_{E_i}(x) \geqq \int_{[0,1]} k\chi_{[0,1]}dx = km([0,1]) = k.$$

The left hand side is

$$\int_{[0,1]}\sum_{i=1}^{n}\chi_{E_i}(x) = \sum_{i=1}^{n} m(E_i).$$

If $m(E_i) < \frac{k}{n}$ for all $i = 1, 2 \cdots n$, then $\sum_{i=1}^{n} m(E_i) < k$ and this contradicts to the result above. So there exists at least one i_0 s.t $m(E_{i_0}) \geqq \frac{k}{n}$.

□

22 (Theorem 4.8) Let $g_k(x) \stackrel{\text{def}}{=} \inf_{m \geqq k}\{f_m(x)\}$. Then $g_k(x) \leqq g_{k+1}(x)$ and $g_k(x) \nearrow \liminf_{k\to\infty} f_k(x)$. By monotone convergence theorem (Theorem 4.4) we have

$$\int_E \lim_{k\to\infty} g_k(x) = \lim_{k\to\infty} \int_E g_k(x)dx$$

And since $g_k(x) \leqq f_k(x)$ we have

$$\int_E g_k(x)dx \leqq \int_E f_k(x)dx$$

This implies that

$$\lim_{k\to\infty} \int_E g_k(x)dx \leqq \liminf_{k\to\infty} \int_E f_k(x)dx$$

- Since $g_k(x)$ is increasing with respect to k, so $\int_E g_k(x)dx$ is also increasing. Therefore $\lim_{k\to\infty} \int_E g_k(x)dx$ exists.
- We do not know if $\lim_{k\to\infty} \int_E f_k(x)$ exists or not. However the $a_n \leqq b_n \Rightarrow \liminf_{n\to\infty} a_n \leqq \liminf_{n\to\infty} b_n$. The left hand side is equal to $\lim_{n\to\infty} a_n$ if the limit exists.

□

23 (Example 5) This example explains that equality does not always hold in Fatou's lemma. First, $f_n(x) \to 0$ for all $x \in [0,1]$ because

- if $x = 0, 1$, $f_n(x) = 0$ for all $n \in \mathbb{N}$ so $\lim_{n\to\infty} f_n(x) = 0$,
- if $x \in (0,1)$, by taking suffiently large n s.t $\frac{1}{n} < x$, $f_n(x) = 0$ so $f_n(x) \to 0$.

So we have

$$\int_{[0,1]} \lim_{n\to\infty} f_n(x)dx = \int_{[0,1]} 0dx = 0$$

However,

$$\int_{[0,1]} f_n(x)dx = \int_{[0,1]} n\chi_{(0,1/n)}(x)dx = nm((0,1/n)) = 1, \ \forall n \in \mathbb{N}.$$

So

$$\lim_{n\to\infty} \int_{[0,1]} f_n(x)dx = 1.$$

□

24 (Theorem 4.9)

STEP 1. Since $f(x) < \infty$ a.e $x \in E$, we may suppose that $f(x) < \infty$ for all $x \in E$ without loss of generarlity. Let $N \stackrel{\text{def}}{=} \{x \in E \mid f(x) = \infty\}$ and let $\tilde{E} \stackrel{\text{def}}{=} E \setminus N$. Since $m(N) = 0$, $\int_E f(x)dx = \int_{\tilde{E}} f(x)dx + \int_N f(x)dx = \int_{\tilde{E}} f(x)dx$. This explains that the integral is determined only on $\tilde{E}$ where $f(x) < \infty$. Therefore we may suppose that $f(x) < \infty$ on E.

STEP 2. Pick a partition $\{y_k\}_{k=0}^{\infty} \in P^{(\delta)}$. Since $f(x) < \infty$ for all $x \in E$ and $E = \bigcup_{k=0}^{\infty} E_k$ and each E_k is mutually disjoint, by Theorem 4.6, we have

$$\int_E f(x)dx = \sum_{k=0}^{\infty} \int_{E_k} f(x)dx$$

On each E_k, $y_k \leqq f(x) < y_{k+1}$, so we have

$$\sum_{k=0}^{\infty} \int_{E_k} y_k dx \leqq \int_E f(x)dx \leqq \sum_{k=0}^{\infty} \int_{E_k} y_{k+1} dx.$$

Hence

$$\sum_{k=0}^{\infty} y_k m(E_k) \leqq \int_E f(x)dx \leqq \sum_{k=0}^{\infty} y_{k+1} m(E_k).$$

Let us take a look at the right hand side.

$$\begin{aligned}
&= \lim_{n\to\infty} \sum_{k=0}^{n} (y_{k+1} - y_k + y_k) m(E_k) \\
&\overset{*1}{\leqq} \lim_{n\to\infty} \sum_{k=0}^{n} (\delta + y_k) m(E_k) \\
&\leqq \lim_{n\to\infty} \left(\sum_{k=0}^{n} y_k m(E_k) + \sum_{k=0}^{n} \delta m(E_k) \right) \\
&\overset{*2}{=} \sum_{k=0}^{\infty} y_k m(E_k) + \delta \sum_{k=0}^{\infty} m(E_k) \\
&\overset{*3}{=} \sum_{k=0}^{\infty} y_k m(E_k) + \delta m(E)
\end{aligned}$$

- $(*1)$ Let us recall that $y_{k+1} - y_k < \delta$.
- $(*2)$ $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ because a_n, b_n are monotone increasing so both limits exist.
- $(*3)$ E_k is mutually disjoint and $E = \bigcup_{k=0}^{\infty} E_k$.

In conclusion we have

$$S(I) \leqq \int_E f(x)dx \leqq S(I) + \delta m(E).$$

From this inequality, we find out that $\int_E f(x) < \infty$ if and only if $S(I) < \infty$. ($\because$ $\delta m(E) < \infty$)

STEP 3. By the inequality in the previous step, we have

$$\sup_{I \in P^{(\delta)}} S(I) \leqq \int_E f(x)dx \leqq \inf_{I \in P^{(\delta)}} S(I) + \delta m(E).$$

By taking limit we have

$$\lim_{\delta \searrow 0} \sup_{I \in P^{(\delta)}} S(I) \leqq \int_E f(x)dx \leqq \lim_{\delta \searrow 0} \inf_{I \in P^{(\delta)}} S(I)$$

This explains that

$$\int_E f(x)dx = \lim_{\delta \searrow 0} \sup_{I \in P^{(\delta)}} S(I) = \lim_{\delta \searrow 0} \inf_{I \in P^{(\delta)}} S(I)$$

□

25 (Example 6) In this question, we use Theorem 4.9 (1).

STEP 1. We pick $\{y_k\}_{k=0}^{\infty}$ where $y_k = k$ in Theorem 4.9. By Theorem 4.9, $\int_E f(x)dx < \infty$ if and only if $\sum_{k=0}^{\infty} y_k m(E_k) < \infty$ where $E_k = \{x \in E \mid k \leqq f(x) < k+1\}$.

STEP 2.

$$\begin{aligned}
\sum_{n=0}^{\infty} m(\{x \in E \mid f(x) \geqq n\}) &\overset{*1}{=} \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} m(\{x \in E \mid k \leqq f(x) < k+1\}) \\
&\overset{*2}{=} \sum_{k=0}^{\infty}\sum_{n=0}^{k} m(\{x \in E \mid k \leqq f(x) < k+1\}) \\
&= \sum_{k=0}^{\infty} k m(\{x \in E \mid k \leqq f(x) < k+1\}) \\
&= \sum_{k=0}^{\infty} y_k m(E_k) < \infty
\end{aligned}$$

if and only if

$$\int_E f(x)dx < \infty.$$

- $(*1)$ by assumption $f(x) < \infty$. $\{x \in E \mid f(x) \geqq n\} = \{x \in E \mid n \leqq f(x) < \infty\} = \bigcup_{k=n}^{\infty}\{x \in E \mid k \leqq f(x) < k+1\}$.
- $(*2)$ if $a_{n,k} \geqq 0$ then $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{n,k} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_{n,k}$. Let $a_{n,k} = \chi_{n \leqq k} \cdot m(\{x \in E \mid k \leqq f(x) < k+1\})$ where $\chi_{n \leqq k} = 1$ if $n \leqq k$, otherwise $= 0$.

□

26 (Example 7) In this question, we use Theorem 4.9 (1) again.

STEP 1. We pick $\{y_k\}_{k=0}^{\infty}$ where $y_k = k^2$ in Theorem 4.9. By Theorem 4.9, $\int_E f^2(x)dx < \infty$ if and only if $\sum_{k=0}^{\infty} y_k m(E_k) < \infty$ where $E_k = \{x \in E \mid k^2 \leqq f^2(x) < (k+1)^2\}$.

STEP 2.

$$\begin{aligned}
\sum_{n=1}^{\infty} nm(\{x \in E \mid f(x) \geqq n\}) &= \sum_{n=0}^{\infty} nm(\{x \in E \mid f(x) \geqq n\}) \\
&= \sum_{n=0}^{\infty} nm(\{x \in E \mid f(x)^2 \geqq n^2\}) \\
&= \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} nm(\{x \in E \mid k^2 \leqq f(x)^2 < (k+1)^2\}) \\
&\stackrel{*1}{=} \sum_{k=0}^{\infty}\sum_{n=0}^{k} nm(\{x \in E \mid k^2 \leqq f(x)^2 < (k+1)^2\}) \\
&\stackrel{*2}{=} \sum_{k=0}^{\infty} \frac{k(k+1)}{2} m(\{x \in E \mid k^2 \leqq f(x)^2 < (k+1)^2\}) \\
&= \sum_{k=0}^{\infty} \frac{k(k+1)}{2} m(E_k) < \infty
\end{aligned}$$

if and only if

$$\sum_{k=0}^{\infty} k^2 m(E_k) \left(= \sum_{k=0}^{\infty} y_k m(E_k) \right) < \infty$$

This is because $k \leqq k^2$ so $\sum_{k=0}^{\infty} k^2 m(E_k) < \infty \Rightarrow \sum_{k=0}^{\infty} km(E_k) < \infty$.

- $(*1)$ Since each term is positive, so we may swap $\sum_n$ and $\sum_k$.
- $(*2)$ $\sum_{n=0}^{k} n = \frac{k(k+1)}{2}$.

□

27 (Exercise 7) Let

$$E_1 \stackrel{\text{def}}{=} \{x \in E \mid 0 \leqq f(x) \leqq 1\}, \text{and } E_2 \stackrel{\text{def}}{=} \{x \in E \mid f(x) > 1\}.$$

By Corollary 4.7

$$\begin{aligned}
\int_E f(x)^2 dx &= \int_{E_1} f(x)^2 dx + \int_{E_2} f^2(x) dx \\
&\leqq \int_{E_1} 1 dx + \int_{E_2} f(x)^3 dx \\
&= m(E_1) + \int_{E_2} f(x)^3 dx \\
&\leqq m(E) + \int_E f(x)^3 dx < \infty.
\end{aligned}$$

- $x \in E_1 \Rightarrow f(x) \leqq 1$.
- $x \in E_2 \Rightarrow f(x)^2 \leqq f(x)^3$ because $f(x) \geqq 1$.

□

28 (Exercise 8) In this question, we use Theorem 4.9 (1) again.

STEP 1. We pick $\{y_k\}_{k=0}^{\infty}$ where $y_k = k^3$ in Theorem 4.9. By Theorem 4.9, $\int_E f^3(x)dx < \infty$ if and only if $\sum_{k=0}^{\infty} y_k m(E_k) < \infty$ where $E_k = \{x \in E \mid k^3 \leqq f^3(x) < (k+1)^3\}$.

STEP 2.

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^2 m(\{x \in E \mid f(x) \geqq n\}) \\
=& \sum_{n=0}^{\infty} n^2 m(\{x \in E \mid f(x) \geqq n\}) \\
=& \sum_{n=0}^{\infty} nm(\{x \in E \mid f(x)^3 \geqq n^3\}) \\
=& \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} n^2 m(\{x \in E \mid k^3 \leqq f(x)^3 < (k+1)^3\}) \\
=& \sum_{k=0}^{\infty}\sum_{n=0}^{k} n^2 m(\{x \in E \mid k^3 \leqq f(x)^3 < (k+1)^3\}) \\
=& \sum_{k=0}^{\infty} \frac{k(k+1)(2k+1)}{6} m(\{x \in E \mid k^3 \leqq f(x)^3 < (k+1)^3\}) \\
=& \sum_{k=0}^{\infty} \frac{k(k+1)(2k+1)}{6} m(E_k) < \infty
\end{aligned}$$

if and only if

$$\sum_{k=0}^{\infty} k^3 m(E_k) \left(= \sum_{k=0}^{\infty} y_k m(E_k) \right) < \infty$$

This is because $k \leqq k^2 \leqq k^3$ so $\sum_{k=0}^{\infty} k^3 m(E_k) < \infty \Rightarrow \sum_{k=0}^{\infty} km(E_k) < \infty$ and $\sum_{k=0}^{\infty} k^2 m(E_k) < \infty$.

□

29 (Exercise 9) Use Fatou's lemma.

STEP 1. By Fatou's lemma

$$\int_e \liminf_{k\to\infty} f_k(x)dx \leqq \liminf_{k\to\infty} \int_e f_k(x)dx.$$

Since $\lim_{k\to\infty} f_k(x) = f(x)$, we have

$$\int_e f(x)dx \leqq \liminf_{k\to\infty} \int_e f_k(x)dx.$$

STEP 2. Since $f_k(x) \leqq f(x)$, $\int_e f_k(x)dx \leqq \int_e f(x)\mathrm{dx}$. Therefore we have

$$\limsup_{k\to\infty} \int_e f_k(x)dx \leqq \int_e f(x)dx$$

Now the proof is complete. □

30 (Exercise 10)

STEP 1. Let us recall that $\limsup_{n\to\infty} E_n = \{x \in [0,1] \mid \#\{n \mid x \in E_n\} = \infty\}$. In otherwords, $\limsup_{n\to\infty} E_n$ is the set of $x \in E$ which is contained by infinitely many $E_n, (n \geqq 1)$. $m(\limsup_{n\to\infty} E_n) = 0$ means that for almost every $x \in [0,1]$, x is contained by only finite number of E_n. Let $f(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \chi_{E_n}(x)$. We can say that $f(x)$ is the number of n s.t $x \in E_n$. By the argument, $f(x) < \infty$ a.e $x \in [0,1]$.

STEP 2. Let $A_m \overset{\text{def}}{=} \{x \in [0,1] \mid f(x) \leqq m\}$. Then $A_m \nearrow \{x \in [0,1] \mid f(x) < \infty\}$. Since A_m is an increasing sequence of point sets (i.e $A_m \subset A_{m+1}$), $\lim_{m\to\infty} m(A_n) = m(\{x \in [0,1] \mid f(x) < \infty\}) = 1$. This implies that we $\forall\epsilon > 0$ we can find sufficiently large M s.t $m(A_M) > 1 - \epsilon$. Therefore $m([0,1] \setminus A_M) < \epsilon$.

STEP 3. Let us consider the integral below.

$$\int_{A_M} f(x)dx.$$

By Theorem 4.6,

$$\begin{aligned}
\int_{A_M} f(x)dx &= \int_{A_M} \sum_{n=1}^{\infty} \chi_{E_n}(x)dx \\
&= \sum_{n=1}^{\infty} \int_{A_M} \chi_{E_n}(x)dx \\
&= \sum_{n=1}^{\infty} m(A_M \cap E_n)
\end{aligned}$$

On the otherhand, $f(x) \leqq M$ on A_M, so we have

$$\int_{A_M} f(x)dx \leqq \int_{A_M} Mdx = Mm(A_M) \leqq M < \infty.$$

So $A \overset{\text{def}}{=} A_M$ is the desired measurable set on $[0,1]$.

□

§ 4.2

31 (Definition: integral of general measurable functions) Let

$$f^+(x) \stackrel{\text{def}}{=} \max\{0, f(x)\} = f(x) \cdot \chi_{\{x \in E | f(x) \geqq 0\}}(x),$$
$$f^-(x) \stackrel{\text{def}}{=} \max\{0, -f(x)\} = f(x) \cdot \chi_{\{x \in E | f(x) \leqq 0\}}(x).$$

Then $f(x) = f^+(x) - f^-(x)$ and $|f(x)| = f^+(x) + f^-(x)$. Let

$$S^+ \stackrel{\text{def}}{=} \int_E f^+(x)dx, \text{ and } S^- \stackrel{\text{def}}{=} \int_E f^-(x)dx.$$

Note that $0 \leqq S^+, S^- \leqq \infty$.

(1) $\int_E f(x)dx \stackrel{\text{def}}{=} S^+ - S^-$. $\int_E f(x)dx$ is defined when at least one of $S^+ < \infty$ or $S^- < \infty$ holds. Then we say that $\int_E f(x)dx$ exists. If $S^+ = \infty, S^- < \infty$, then $\int_E f(x)dx = \infty$, and if $S^+ < \infty, S^- = \infty$, then $\int_E f(x)dx = -\infty$.

(2) When both $S^+, S^- < \infty$, we say that $f(x)$ is integrable.

(3) $\int_E |f(x)|dx = \int_E f^+(x)dx + \int_E f^-(x)dx = S^+ + S^-$ by Corollary 4.7. So

$$\begin{aligned} \int_E |f(x)|dx < \infty &\Leftrightarrow S^+ + S^- < \infty \\ &\Leftrightarrow S^+, S^- < \infty \\ &\Leftrightarrow f(x) \text{ is integrable.} \end{aligned}$$

(4) $\left|\int_E f(x)dx\right| = |S^+ - S^-| \leqq S^+ + S^- = \int_E |f(x)|dx$.

□

32 (Example 1) Let us recall that f is integrable if and only if $|f|$ is integrable. If f is bounded then $|f| \leqq M < \infty$ for some $M > 0$. $\int_E |f(x)|dx \leqq \int_E M dx = M \cdot m(E) < \infty$. ($\because m(E) < \infty$ by assumption.) □

33 (Some Properties)

(1) $f(x) \in L(E)$ means that $f(x)$ is integrable $\Leftrightarrow |f(x)|$ is integrable. If not $|f(x)| < \infty$ a.e $x \in E$, then $m(\{x \in E \mid |f(x)| = \infty\}) > 0$. Then, we have $\int_E |f(x)|dx \geqq \int_{\{x \in E || f(x)| = \infty\}} |f(x)|dx \geqq \int_{\{x \in E || f(x)| = \infty\}} \infty dx = \infty \cdot m(\{x \in E \mid |f(x)| = \infty\}) = \infty$. (contradiction!!) So $|f(x)| < \infty$ a.e $x \in E$ holds.

(2) $f(x) = 0$ a.e $x \in E \Leftrightarrow |f(x)| = 0$ a.e $x \in E \Rightarrow \int_E |f(x)|dx = 0$ (See §4.1 Properties of integral of non-negative measurable functions). And $0 = \int_E |f(x)|dx \geqq \left|\int_E f(x)dx\right|$. So $\int_E f(x)dx = 0$.

(3) See §4.1 Properties of integral of non-negative measurable functions. From the assumption, we find out that $|f(x)|$ is integrable. So $f(x)$ is integrable.

(4) Let $E \stackrel{\text{def}}{=} \mathbb{R}^d$ and let $f_k(x) \stackrel{\text{def}}{=} |f(x)| \cdot \chi_{\{x \in E | |x| \geqq k\}}$. Then $0 \leqq f_k(x) \leqq |f(x)|$ so $f_k(x)$ is integrable. And $\{f_k(x)\}_{k \geqq 1}$ is a decreasing sequence of integrable functions (i.e $f_{k+1}(x) \leqq f_k(x)$), and

$$f_k(x) \to 0, \forall x \in E,$$

because $\forall x \in E$, by taking sufficiently large $k \in \mathbb{N}$, we have $|x| < k$. By §4.1 Example 2, we have

$$\lim_{k \to \infty} \int_E |f_k(x)| dx = \int_E 0 \ dx = 0.$$

By §4.1 properties of integral,

$$\int_E |f_k(x)| dx = \int_{\{x \in E | |x| \geqq k\}} |f(x)| dx.$$

Now the proof is comlete.

□

34 (Theorem 4.10 Linearity of Lebesgue Integral)

(1)

case 1. $(C > 0)$

$$\begin{aligned} \int_E Cf(x)dx &\stackrel{*1}{=} \int_E (Cf)^+(x)dx - \int_E (Cf)^-(x)dx \\ &\stackrel{*2}{=} \int_E Cf^+(x)dx - \int_E Cf^-(x)dx \\ &\stackrel{*3}{=} C\int_E f^+(x)dx - C\int_E f^-(x)dx = C\int_E f(x)dx \end{aligned}$$

- $(*1)$ By definition.
- $(*2)$ $C > 0$. So $(Cf)^+ = C(f^+)$.
- $(*3)$ See Theorem 4.5.

case 2. $(C = 0)$ Obvious.

case 3. $(C < 0)$ Repeat the similar argument. But note that $(Cf)^+ = -Cf^-$,

$(Cf)^- = -Cf^+$.

$$\begin{aligned}\int_E Cf(x)dx &= \int_E (Cf)^+(x)dx - \int_E (Cf)^-(x)dx \\ &= \int_E (-C)\cdot f^-(x)dx - \int_E (-C)f^+(x)dx \\ &\overset{*4}{=} (-C)\cdot \int_E f^-(x)dx - (-C)\int_E f^+(x)dx \\ &= -C\cdot \int_E f^-(x)dx + C\cdot \int_E f^+(x)dx \\ &= C\left(\int_E f^+(x)dx - \int_E f^-(x)dx\right) \\ &= C\cdot \int_E f(x)dx\end{aligned}$$

- (*4) Recall that $\int_E \alpha f(x)dx = \alpha \int_E f(x)dx$ if $\alpha > 0$ and $f(x)$ is a non-negative measurable function.

(2) $f(x) \in L(E)$ and $\int_E g(x)dx$ exists. So $\int_E f^+(x)dx, \int_E f^-(x)dx < \infty$, and at least one of $\int_E g^+(x)dx < \infty$ or $\int_E g^-(x)dx < \infty$ holds. Let

$$h(x) \overset{\text{def}}{=} f(x) + g(x).$$

By separating each function to a positive part and a negative part, we have

$$h^+ - h^- = f^+ - f^- + g^+ - g^-,$$

hence

$$h^+ + f^- + g^- = h^- + f^+ + g^+$$

So we have

$$\int_E (h^+ + f^- + g^-)dx = \int_E (h^- + f^+ + g^+)dx,$$

and by Theorem 4.5, (we sometimes omit dx)

$$\int_E h^+ + \int_E f^- + \int_E g^- = \int_E h^- + \int_E f^+ + \int_E g^+.$$

case 1. ($\int_E g^- < \infty$) $h^- \leqq f^- + g^-$ so $\int_E h^- < \infty$. Since we may subtract finite terms ($\int_E h^-, \int_E f^-, \int_E g^-$) from both sides, we have

$$\begin{aligned}&\int_E h^+ + \int_E f^- + \int_E g^- - \int_E h^- - \int_E f^- - \int_E g^- \\ = \ &\int_E h^- + \int_E f^+ + \int_E g^+ - \int_E h^- - \int_E f^- - \int_E g^-,\end{aligned}$$

and this implies that

$$\begin{aligned}&\int_E h^+ - \int_E h^- \\ = \ &\int_E f^+ + \int_E g^+ - \int_E f^- - \int_E g^-,\end{aligned}$$

so we have

$$\int_E h = \int_E f + \int_E g.$$

case 2. ($\int_E g^+ < \infty$) Similarly, $h^+ \leqq f^+ + g^+$ so $\int_E h^+ < \infty$, and by subtracting them from the both sides like the previous step, we obtain

$$\begin{aligned} & \int_E h^+ + \int_E f^- + \int_E g^- - \int_E h^+ - \int_E f^+ - \int_E g^+ \\ = & \int_E h^- + \int_E f^+ + \int_E g^+ - \int_E h^+ - \int_E f^+ - \int_E g^+, \end{aligned}$$

and this implies that

$$\begin{aligned} & \int_E h^- - \int_E h^+ \\ = & \int_E f^- + \int_E g^- - \int_E f^+ - \int_E g^+, \end{aligned}$$

so we have

$$-\int_E h = -\int_E f - \int_E g,$$

and we have the desired result by multiplying -1 to the both sides.

□

35 (Example 2) We separate $[0,1]$ into $E_1 \overset{\text{def}}{=\!=} \{x \in [0,1] \mid |f(x)| > e-1\}$ and $E_2 \overset{\text{def}}{=} \{x \in [0,1] \mid |f(x)| \leqq e-1\}$. $|f(x)| \ln(1+|f(x)|)$ is non-negative.

$$\begin{aligned} \int_{[0,1]} |f(x)| dx &= \int_{E_1} |f(x)| dx + \int_{E_2} |f(x)| dx && (4.1)\\ &\leqq \int_{E_1} |f(x)| \ln(1+|f(x)|)\, dx + \int_{E_2} (e-1) dx && (4.2)\\ &\leqq \int_{[0,1]} |f(x)| \ln(1+|f(x)|)\, dx + \int_{[0,1]} (e-1) dx < \infty && (4.3)\\ & && (4.4) \end{aligned}$$

□

36 (Example 3) Let $g_n(x) \overset{\text{def}}{=} f_n(x) - f_1(x) \geqq 0$. $\{g_n(x)\}_{n \geqq 1}$ is an increasing sequence of non-negative measurable functions. By monotone convergence theorem, we have

$$\lim_{n\to\infty} \int_E g_n(x) dx = \int_E (f(x) - f_1(x)) dx$$

Since $f_1 \in L(E)$,

$$\begin{aligned} \lim_{n\to\infty} \int_E g_n(x)dx &= \lim_{n\to\infty} \left(\left(\int_E f_n(x)dx - \int_E f_1(x) \right) dx \right. \\ &= \lim_{n\to\infty} \int_E f_n(x)dx - \int_E f_1(x)dx \end{aligned}$$

and

$$\int_E (f(x) - f_1(x))dx = \int_E f(x)dx - \int_E f_1(x)dx.$$

$\int_E f_1(x)dx$ is finite, so we may add $\int_E f_1(x)dx$ to the both sides. Then we have the desired result. (Notice) The original textbook gives an assumption $f \in L(E)$ however we do not need to assume that $f \in L(E)$. □

37 (Example 4) Let $g_n(x) \stackrel{\text{def}}{=} f_n(x) - g(x) \geqq 0$. Since $\{g_n(x)\}_{n\geqq 1}$ is a sequence of non-negative measurable function, so we can apply Fatou's lemma to g_n.

$$\int_E \liminf_{n\to\infty} g_n(x)dx \leqq \liminf_{n\to\infty} \int_E g_n(x)dx$$

And

$$\begin{aligned} \int_E \liminf_{n\to\infty} g_n(x)dx &= \int_E \liminf_{n\to\infty} (f_n(x) - g(x))dx \\ &= \int_E (\liminf_{n\to\infty} f_n(x) - g(x))dx \\ &\stackrel{*}{=} \int_E \liminf_{n\to\infty} f_n(x)dx - \int_E g(x)dx \end{aligned}$$

$$\liminf_{n\to\infty} \int_E g_n(x)dx \stackrel{*}{=} \liminf_{n\to\infty} \int_E f_n(x)dx - \int_E g(x)dx$$

- (*) $g(x)$ is integrable. See Theorem 4.10.
- Finally, since $\int_E g(x)dx$ is finite, we can add it to the both sides.

□

38 (Example 5) □

39 (Exercise 1)

$$\begin{aligned} -(f^- + g^-) &\leqq \min\{f(x), g(x)\} \\ &\leqq \max\{f(x), g(x)\} \leqq f^+(x) + g^+(x). \end{aligned}$$

So $|m(x)|, |M(x)| \leqq |f(x)| + |g(x)| \in L(E)$. □

40 (Exercise 2)

STEP 1. Since $xy \notin \mathbb{Q}$ a.e $(x, y) \in [0,1] \times [0,1]$, $(*)$,

$$\int_{[0,1]\times[0,1]} f(x)dx = 1.$$

STEP 2. (proof of $(*)$) We prove that

$$m(\{(x, y) \in [0,1] \times [0,1] \mid xy \in \mathbb{Q}\}) = 0.$$

Let us consider a curve (or a line if $r = 0$) $C_r \overset{\text{def}}{=} \{(x, y) \in [0,1] \times [0,1] \mid xy = r\}$ where $r \in \mathbb{Q} \cap [0,1]$. It is enough for us to show that $m(C_r) = 0$.

case 1. $(r = 0)$ Lines $\{0\} \times [0,1]$ and $[0,1] \times \{0\}$ have measure zero. For all $\epsilon > 0$, $\{0\} \times [0,1] \subset (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times [0,1]$. $m((-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times [0,1]) = \epsilon$.

case 2. $(r > 0)$ We cover a curve C_r by n rectangles. Let us pick $n+1$ points $\{x_i\}_{i=0}^n$ where $r = x_0 < x_1 < x_2 < \cdots < x_n = 1, x_i - x_{i-1} = \frac{1-r}{n}$. Let us consider rectangles $I_i \overset{\text{def}}{=} [x_{i-1}, x_i] \times [\frac{r}{x_i}, \frac{r}{x_{i-1}}]$, $i = 1 \cdots n$. Then $m(I_i) = m([x_{i-1}, x_i] \times [\frac{r}{x_i}, \frac{r}{x_{i-1}}]) = \frac{r(x_{i-1}-x_i)^2}{x_{i-1}x_i}$. Since $C_r \subset \cup_{i=1}^n I_i$, we have

$$m^*(C_r) \leqq \sum_{i=1}^n \frac{r(x_i - x_{i-1})^2}{x_{i-1}x_i}.$$

Moreover $(x_i - x_{i-1}) = \frac{1-r}{n}$ and $r \leqq x_0 \leqq \cdots \leqq x_n$, therefore,

$$m^*(C_r) \leqq \sum_{i=1}^n \frac{r(x_i - x_{i-1})^2}{x_{i-1}x_i} \leqq \sum_{i=1}^n \frac{r(1-r)^2}{n^2 r^2} = \frac{(1-r)^2}{nr}.$$

This holds for all $n \in \mathbb{N}$, so by taking $n \to \infty$, we have

$$m^*(C_r) = 0.$$

□

41 (Exercise 3) We show that

$$\lim_{k\to\infty} \frac{m\left(\{x \in E \mid |f(x)| > k\}\right)}{\frac{1}{k}} \to 0.$$

So, we prove that

$$\lim_{k\to\infty} k \cdot m\left(\{x \in E \mid |f(x)| > k\}\right) \to 0.$$

First,

$$\begin{aligned} k \cdot m\left(\{x \in E \mid |f(x)| > k\}\right) &= \int_E k\chi_{\{x\in E||f(x)|>k\}}(x)dx \\ &\overset{*}{\leqq} \int_E |f(x)|\chi_{\{x\in E||f(x)|>k\}}(x)dx. \end{aligned}$$

- $(*)$ $k < |f(x)|$ if $\chi_{\{x\in E||f(x)|>k\}}(x) = 1$.

Let

$$f_k(x) \stackrel{\text{def}}{=} |f(x)|\chi_{\{x\in E||f(x)|>k\}}(x).$$

Then $f_k(x)$ is a decreasning sequence of integrable functions. (i.e $f_{k+1}(x) \leqq f_k(x)$.) Moreover $f_k(x) \to \infty \cdot \chi_{\{x\in E||f(x)|=\infty\}} = 0$ a.e $x \in E$ because $f(x)$ is integrable so $|f(x)| < \infty$ a.e $x \in E$. So we conclude that

$$f_k(x) \xrightarrow{\text{a.e}} 0 \text{ on } E.$$

By §4.1,Example 2, we have

$$\lim_{k\to\infty} \int_E f_k(x)dx = \int_E 0\ dx = 0.$$

□

42 (Exercise 4)

STEP 1. Let $\epsilon > 0$. $\lim_{n\to\infty} m(\{x \in E \mid |f_n(x) - f(x)| > \epsilon\}) = 0$ if and only if $\lim_{n\to\infty} \epsilon m(\{x \in E \mid |f_n(x) - f(x)| > \epsilon\}) = 0$

STEP 2.

$$\begin{aligned}
& \limsup_{n\to\infty} \epsilon m\left(\{x \in (0,\infty) \mid |f_n(x) - f(x)| > \epsilon\}\right) \\
= \ & \limsup_{n\to\infty} \epsilon m\left(\{x \in (0,\infty) \mid |f(x)| \cdot \chi_{[n,\infty)}(x) > \epsilon\}\right) \\
= \ & \limsup_{n\to\infty} \epsilon m\left(\{x \in [n,\infty) \mid |f(x)| > \epsilon\}\right) \\
= \ & \limsup_{n\to\infty} \int_{(0,\infty)} \epsilon \cdot \chi_{\{x\in[n,\infty)||f(x)|>\epsilon\}}(x)dx \\
\leqq \ & \limsup_{n\to\infty} \int_{(0,\infty)} |f(x)| \cdot \chi_{\{x\in[n,\infty)||f(x)|>\epsilon\}}(x)dx \\
\stackrel{*1}{\leqq} \ & \limsup_{n\to\infty} \int_{(0,\infty)} |f(x)| \cdot \chi_{\{x\in[n,\infty)\}}(x)dx \stackrel{*2}{=} 0
\end{aligned}$$

- $(*1)$ We get rid of $|f(x)| > \epsilon$ from the indicator function $\chi_{\{\cdots\}}$. This means that we give a weaker condition for the indicator function to be 1. (Hence greater.)

- $(*2)$ This is similar to the previous question. $f_n(x) \stackrel{\text{def}}{=} |f(x)|\chi_{\{x\in[n,\infty)\}}(x)$ is integrable for all $n \in \mathbb{N}$, $f_{n+1}(x) \leqq f_n(x)$ and $f_n(x) \to 0$ for all $x \in (0,\infty)$. So By §4.1 Example 2 we have the desired conclusion.

□

43 (Exercise 5)

STEP 1. Let us recall that if $f(x)$ is a non negative measurable function with $\int_E f(x)dx = 0$ then $f(x) = 0$ a.e $x \in E$.

STEP 2. $C = \int_{[0,1]} f(x)dx$. By the hint,

$$e^C(f(x) - C) + e^C \leqq e^{f(x)} \Rightarrow e^{f(x)} - e^C(f(x) - C) - e^C \geqq 0,$$

and the equality holds if $f(x) = C$. Since $e^{f(x)} - e^C(f(x) - C) - e^C$ is a non-negative measurable function, we have

$$\int_{[0,1]} \left(e^{f(x)} - e^C(f(x) - C) - e^C\right) \geqq 0.$$

By assumption, $f(x)$ is integrable,

$$\begin{aligned} \int_{[0,1]} \left(e^{f(x)} - e^C(f(x) - C) - e^C\right) &= \int_{[0,1]} e^{f(x)}dx - e^C \int_{[0,1]} f(x)dx + C \cdot e^C - e^C \\ &\overset{*1}{=} \int_{[0,1]} e^{f(x)}dx - C \cdot e^C + C \cdot e^C - e^C \\ &= \int_{[0,1]} e^{f(x)}dx - e^C \overset{*2}{=} 0 \end{aligned}$$

- $(*1)$ recall that $C = \int_{[0,1]} f(x)dx$
- $(*2)$ by assumption

By the statement in Step 1, we conclude that

$$e^{f(x)} - e^C(f(x) - C) - e^C = 0 \text{ a.e } x \in [0,1].$$

The equality holds $f(x) = C$. So $f(x) = C$ a.e $x \in [0,1]$.

□

44 (Exercise 6) We use Theorem 4.11. (Please see Theorem 4.11.)

STEP 1. Since $I = E_I \cup I \setminus E_I$, by Theorem 4.6,

$$\int_I |f(x) - f_I|dx = \int_{E_I} |f(x) - f_I|dx + \int_{I \setminus E_I} |f(x) - f_I|dx.$$

And if $x \in E_I$, $f(x) - f_I > 0$, so

$$\int_{E_I} |f(x) - f_I|dx = \int_{E_I} (f(x) - f_I)dx.$$

Therefore,

$$\int_I |f(x) - f_I|dx = \int_{E_I} (f(x) - f_I)dx + \int_{I \setminus E_I} |f(x) - f_I|dx.$$

It is enough for us to show that

$$\int_{I \setminus E_I} |f(x) - f_I|dx = \int_{E_I} (f(x) - f_I)dx.$$

STEP 2. Since $f(x) - f_I \leqq 0$, we have

$$\int_{I \setminus E_I} |f(x) - f_I| dx = \int_{I \setminus E_I} (f_I - f(x)) dx.$$

Next, $f_I - f(x)$ is integrable on I because $f(x) \in L(\mathbb{R}^1)$, so $f(x) \in L(I)$ and $|f_I - f(x)| \leqq |f_I| + |f(x)| \in L(I)$. ($I$ is bounded.) By Theorem 4.11,

$$\int_{E_I} (f_I - f(x)) dx + \int_{I \setminus E_I} (f_I - f(x)) dx = \int_I (f_I - f(x)) dx.$$

(Let us pay attention to the fact that the both terms on the left side are also integrable because $I, I \setminus E_I$ are the subsets of I.) And

$$\begin{aligned} \int_I (f_I - f(x)) dx &= |I| \cdot f_I - \int_I f(x) dx \\ &= |I| \cdot \frac{1}{|I|} \int_I f(x) dx - \int_I f(x) dx = 0. \end{aligned}$$

So

$$\int_{E_I} (f_I - f(x)) dx + \int_{I \setminus E_I} (f_I - f(x)) dx = 0.$$

Therefore, (Theorem 4.10)

$$\int_{I \setminus E_I} (f_I - f(x)) dx = -\int_{E_I} (f_I - f(x)) dx = \int_{E_I} (f(x) - f_I) dx.$$

Now the proof is complete.

□

45 (Theorem 4.11) By definition,

$$\int_E f(x) dx = \int_E f^+(x) dx - \int_E f^-(x) dx.$$

Note that $\int_E f(x) dx$ exists implies that $\int_E f^+(x) dx < \infty$ or $\int_E f^-(x) dx < \infty$ holds. By Corollary 4.7, we have

$$\begin{aligned} \int_E f^+(x) dx &= \sum_{k=1}^{\infty} \int_{E_k} f^+(x) dx \\ &= \lim_{n \to \infty} \sum_{k=1}^{n} \int_{E_k} f^+(x) dx. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_E f^-(x) dx &= \sum_{k=1}^{\infty} \int_{E_k} f^-(x) dx \\ &= \lim_{n \to \infty} \sum_{k=1}^{n} \int_{E_k} f^-(x) dx. \end{aligned}$$

Let

$$a_n \overset{\text{def}}{=} \sum_{k=1}^{n} \int_{E_k} f^+(x)dx \text{ and } b_n \overset{\text{def}}{=} \sum_{k=1}^{n} \int_{E_k} f^-(x)dx,$$

then a_n, b_n are monotone increasing, and $a_n \nearrow \sum_{k=1}^{\infty} \int_{E_k} f^+(x)dx$, $b_n \nearrow \sum_{k=1}^{\infty} \int_{E_k} f^-(x)dx$. Since $\lim_{n\to\infty} a_n < \infty$ or $\lim_{n\to\infty} b_n < \infty$,

$$\lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = \lim_{n\to\infty} (a_n - b_n).$$

Therefore

$$\begin{aligned}
\int_E f(x)dx &= \int_E f^+(x)dx - \int_E f^-(x)dx \\
&= \sum_{k=1}^{\infty} \int_{E_k} f^+(x)dx - \sum_{k=1}^{\infty} \int_{E_k} f^-(x)dx \\
&= \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n \\
&= \lim_{n\to\infty} (a_n - b_n) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \int_{E_k} f^+(x)dx - \sum_{k=1}^{n} \int_{E_k} f^-(x)dx \right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\int_{E_k} f^+(x)dx - \int_{E_k} f^-(x)dx \right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\int_{E_k} (f^+(x)dx - f^-(x))dx \right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\int_{E_k} f(x)dx \right) = \sum_{k=1}^{\infty} \int_{E_k} f(x)dx
\end{aligned}$$

□

46 (Example 6) We can easily find out that $\forall a_i, b_i \in [a,b]$, we have $\int_{(a_i,b_i)} f(x)dx = 0$. By assumption $\int_{[a,b_i]} f(x)dx = 0$, $\int_{[a,a_i]} f(x)dx = 0$. Since they are integrable, we have $\int_{[a,b_i]} f(x)dx - \int_{[a,a_i]} f(x)dx = \int_{(a_i,b_i]} f(x)dx = \int_{(a_i,b_i)} f(x)dx$. ($\because m(\{b_i\}) = 0$.)

STEP 1. In this question, we consider the contraposition. Suppose $m(\{x \in [a,b] \mid f(x) \neq 0\}) > 0$. Since $m(\{a\}), m(\{b\}) = 0$, $m(\{x \in (a,b) \mid f(x) \neq 0\}) > 0$. At least $m(\{x \in (a,b) \mid f(x) > 0\}) > 0$ or $m(\{x \in (a,b) \mid f(x) < 0\}) > 0$ holds. We suppose that $m(\{x \in (a,b) \mid f(x) > 0\}) > 0$.

Let $A \overset{\text{def}}{=} \{x \in (a,b) \mid f(x) > 0\}$. $0 < m(A) \leqq b - a$ and $A \in \mathscr{M}$. So there exists F : a closed set, s.t $F \subset A, m(A \setminus F) < \epsilon = m(A)$. Since $m(A) < \infty$, $m(A \setminus F) = m(A) - m(F) < m(A)$, $m(F) > 0$.

STEP 2. Suppose $\int_F f(x)dx = 0$. Since $f(x)$ is non-negative on F, $f(x) = 0$ a.e $x \in F$ by properties of integral of non-negative measurable functions. However, $f(x) > 0$ on F and $m(F) > 0$, so $f(x) = 0$ a.e $x \in F$ does not hold. (contradiction!!) Therefore $\int_F f(x)dx > 0$. (Moreover $f(x) \in L([a,b])$ so $\int_F f(x) < \infty$.)

Let $G = (a,b) \setminus F$. Then $(a,b) = F \cup G$. $\int_{(a,b)} f(x)dx = \int_F f(x)dx + \int_G f(x)dx = 0$. So $\int_G f(x)dx < 0$.

STEP 3. Since G is an open set, there exist disjoint open intervals $\{(a_n, b_n)\}_{n=1}^{\infty}$ s.t $G = \bigcup_{n=1}^{\infty}(a_n, b_n)$. $\int_G f(x)dx = \sum_{n=1}^{\infty} \int_{(a_n,b_n)} f(x)dx = 0$. This contradicts to the conclusion of Step 2.

□

47 (Example 7) $g(x)$ is bounded a.e $x \in E$ means that $\exists k < \infty$ s.t $m(\{x \in E \mid |g(x)| > k\}) = 0$. We suppose that $g(x)$ is bounded a.e $x \in E$ does NOT hold, and derive a contradiction. In other words, we suppose that

$$\forall k \in \mathbb{N}, m(\{x \in E \mid |g(x)| > k\}) > 0.$$

STEP 1. We claim that there exists a subsequence $\{k_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ s.t $m(\{x \in E \mid k_i < |g(x)| \leqq k_{i+1}\}) > 0$. By assumption,

$$\begin{aligned} m(\{x \in E \mid |g(x)| > k\}) &\overset{*1}{=} m(\{x \in E \mid k < |g(x)| < \infty\}) \\ &= m\left(\bigcup_{i=1}^{\infty}\{x \in E \mid k < |g(x)| \leqq k+i\}\right) \\ &\overset{*2}{=} \lim_{i\to\infty} m(\{x \in E \mid k < |g(x)| \leqq k+i\}) > 0 \end{aligned}$$

- (∗1) $g(x) : E \mapsto \mathbb{R}$ by assumption.
- (∗2) $\{x \in E \mid k < |g(x)| \leqq k+i\}$ is increasing with respect to i. So we can swap m and lim.

This means that there exists i_0 s.t $m(\{x \in E \mid k < |g(x)| \leqq k+i_0\}) > 0$. Next $m(\{x \in E \mid |g(x)| > k+i_0\}) > 0$ by assumption. By repeating the similar argument, we have i_1 s.t $m(\{x \in E \mid k_{i_0} < |g(x)| \leqq k+i_0+i_1\}) > 0$.

STEP 2. Let $E_i \overset{\text{def}}{=} \{x \in E \mid k_i < |g(x)| \leqq k_{i+1}\}$ We define

$$f(x) \overset{\text{def}}{=} \sum_{i=1}^{\infty} \frac{1}{i^{3/2} \cdot m(E_i)} \chi_{E_i}(x).$$

Then

$$\int_E f(x)dx \overset{*3}{=} \sum_{i=1}^{\infty} \int_E \frac{1}{i^{3/2} \cdot m(E_i)} \chi_{E_i}(x)dx = \sum_{i=1}^{\infty} \frac{1}{i^{3/2}} < \infty$$

- (∗3) Use Theorem 4.6.

However,

$$\begin{aligned}
\int_E |f(x)g(x)|dx &= \int_E |g(x)| \cdot \sum_{i=1}^{\infty} \frac{1}{i^{3/2} \cdot m(E_i)} \chi_{E_i}(x)dx \\
&= \int_E \sum_{i=1}^{\infty} \frac{|g(x)|}{i^{3/2} \cdot m(E_i)} \chi_{E_i}(x)dx \\
&\overset{*4}{\geqq} \int_E \sum_{i=1}^{\infty} \frac{k_i}{i^{3/2} \cdot m(E_i)} \chi_{E_i}(x)dx \\
&\overset{*5}{\geqq} \int_E \sum_{i=1}^{\infty} \frac{i}{i^{3/2} \cdot m(E_i)} \chi_{E_i}(x)dx \\
&= \int_E \sum_{i=1}^{\infty} \frac{1}{i^{1/2} \cdot m(E_i)} \chi_{E_i}(x)dx \\
&\overset{*6}{=} \sum_{i=1}^{\infty} \int_E \frac{1}{i^{1/2} \cdot m(E_i)} \chi_{E_i}(x)dx \\
&= \sum_{i=1}^{\infty} \frac{1}{i^{1/2}} = \infty,
\end{aligned}$$

so we have $|f(x)g(x)| \notin L(E)$. (contradiction!!)

- $(*4)$ $g(x) > k_i$ on E_i.
- $(*5)$ $k_i \geqq i$ because it is a subsequence of natural numbers.
- $(*6)$ Theorem 4.6.

□

48 (Theorem 4.12)

STEP 1. Let $g_n(x) \overset{\text{def}}{=} |f(x)|\chi_{\{x\in||f(x)|>n\}}(x)$. $g_n(x)$ is a decreasing sequence of integrable functions. $\lim_{n\to\infty} g_n(x) = \infty \cdot \chi_{\{x\in||f(x)|=\infty\}}(x)$. However, $f(x) \in L(E)$, so $|f(x)| < \infty$ a.e $x \in E$ by Theorem 4.3. (i.e $m(\{x \in E \mid |f(x)| = \infty\}) = 0$.) Therefore $\lim_{n\to\infty} g_n(x) = 0$ a.e $x \in E$. By Example 2 in §4.1, we have $\lim_{n\to\infty} \int_E g_n(x)dx = 0$. For any $\epsilon > 0$, we have sufficiently large n_0 s.t $\int_E g_{n_0}(x)dx < \frac{\epsilon}{2}$.

STEP 2. Let $A \subset E, A \in \mathscr{M}$ be an arbitrary measurable subset of E with $m(A) <$

$\frac{\epsilon}{2n_0}$.

$$\begin{aligned}
\left|\int_A f(x)dx\right| &\leqq \int_A |f(x)|dx \\
&\overset{*1}{=} \int_{\{x\in A||f(x)|>n_0\}} |f(x)|dx + \int_{\{x\in A||f(x)|\leqq n_0\}} |f(x)|dx \\
&\overset{*2}{\leqq} \int_{\{x\in E||f(x)|>n_0\}} |f(x)|dx + \int_{\{x\in A||f(x)|\leqq n_0\}} n_0 dx \\
&\overset{*3}{=} \int_E |f(x)| \cdot \chi_{\{x\in E||f(x)|>n_0\}} dx + n_0 m(\{x \in A \mid |f(x)| \leqq n_0\})dx \\
&= \int_E g_{n_0}(x)dx + n_0 m(\{x \in A \mid |f(x)| \leqq n_0\})dx \\
&\overset{*4}{\leqq} \int_E g_{n_0}(x)dx + n_0 m(A) < \frac{\epsilon}{2} + \frac{\epsilon}{2}
\end{aligned}$$

- ($*1$) divide A into two disjoint measurable sets.
- ($*2$) $A \subset E$ and $x \in \{x \in A \mid |f(x)| \leqq n_0\} \Rightarrow |f(x)| \leqq n_0$.
- ($*3$) apply properties of integral in §4.1 ; integral of a simple function
- ($*4$) $\{x \in A \mid |f(x)| \leqq n_0\} \subset A$.

□

49 (Example 8)

STEP 1. Let $E_t \overset{\text{def}}{=} E \cap (-\infty, t) \in \mathscr{M}$. Let $g(t) \overset{\text{def}}{=} \int_{E_t} f(x)dx$. Since $f(x) \in L(E)$, $g(t)$ is well-defined and finite. We show that $g(t)$ is a continuous function.

$$\begin{aligned}
g(t+\Delta t) - g(t) &= \int_{E_{t+\Delta t}} g(x)dx - \int_{E_t} g(x)dx \\
&= \int_{E_{t+\Delta t}\setminus E_t} g(x)dx \\
&= \int_{E\cap[t,t+\Delta t)} g(x)dx
\end{aligned}$$

Since $m(E \cap [t, t + \Delta t)) \leqq m([t, t + \Delta t)) = \Delta t$, by Theorem 4.12, if $\Delta t \searrow 0$, $g(t + \Delta t) - g(t) \to 0$ for all $t \in \mathbb{R}$. So we conclude that $g(t)$ is continuous.

STEP 2. Next, we show that $\lim_{t\to\infty} g(t) = \int_E f(x) = A$. $\lim_{t\to-\infty} g(t) = 0$.

Since $g(t) = \int_{E_t} f(x)dx = \int_E f(x) \cdot \chi_{E_t}(x)dx$, and $f(x) \cdot \chi_{E_t}$ is monotone increasing with respect to t, $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \int_E f(x) \cdot \chi_{E_t}(x)dx = \int_E f(x)dx = A$ by monotone convergence theorem. And $\lim_{t\to-\infty} g(t) = \lim_{t\to-\infty} \int_E f(x) \cdot \chi_{E_t}(x)dx = \int_E 0\ dx = 0$ by Example 2 in §4.1. (Note that $E_t \searrow \emptyset$ as $t \to -\infty$. Let us pick an arbitrary point $x_0 \in \mathbb{R}$. If t is sufficiently small, $x_0 \notin (-\infty, t)$. This implies that $(-\infty, t) \searrow \emptyset$ as $t \to -\infty$, hence $E_t \searrow \emptyset$ because $E_t \subset (-\infty, t)$.)

STEP 3. By intermediate value theorem, we can find t_0 s.t $g(t_0) = \frac{A}{3}$. So $e \stackrel{\text{def}}{=} E_{t_0} = E \cap (-\infty, t_0)$ is the desired subset of E.

□

50 (Theorem 4.13)

STEP 1. (non-negative measurable simple function) Let $f(x) \stackrel{\text{def}}{=} \sum_{i=1}^{p} a_i \chi_{E_i}(x)$ where $E_i \in \mathscr{M}, a_i \geqq 0$. Then $\int_{\mathbb{R}^d} f(x)dx = \sum_{i=1}^{p} a_i m(E_i)$. And $f(x+y_0)dx = \sum_{i=1}^{p} a_i \chi_{E_i}(x+y_0) = \sum_{i=1}^{p} a_i \chi_{E_i - y_0}(x)$. Let us recall that we proved that $\forall a \in \mathbb{R}^d$ and $E \in \mathscr{M}; E \subset \mathbb{R}^d$, $E_{+a} \in \mathscr{M}$ and $m(E_{+a}) = m(E)$ in Theorem 2.5. Therefore $f(x+y_0)$ is also a non-negative measurable simple function and $\int_{\mathbb{R}^d} f(x+y_0)dx = \sum_{i=1}^{p} a_i m(E_{i-y_0}) = \sum_{i=1}^{p} a_i m(E_i)$. So $\int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} f(x+y_0)dx$.

STEP 2. (non-negative measurable function) Let $f(x)$ be a non-negative measurable function. We can find a sequence of non-negative measurable simple functions $\{f_n(x)\}$ s.t $f_n(x) \nearrow f(x)$ for all $x \in \mathbb{R}^d$ by Theorem 3.9. So $\lim_{n\to\infty} f_n(x+y_0) = f(x+y_0)$. By Theorem 4.4 (monotone convergence theorem) and the previous result, $\int_{\mathbb{R}^d} f(x+y_0)dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x+y_0)dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx = \int_{\mathbb{R}^d} f(x)$.

STEP 3. (measurable function) By the previous result, $\int_{\mathbb{R}^d} f^+(x)dx = \int_{\mathbb{R}^d} f^+(x+y_0)dx$ and $\int_{\mathbb{R}^d} f^-(x)dx = \int_{\mathbb{R}^d} f^-(x+y_0)dx$. $\int_{\mathbb{R}^d} f(x)dx$ exists. At least one of $\int_{\mathbb{R}^d} f^+(x)dx$, $\int_{\mathbb{R}^d} f^-(x)dx$ is finite. So we are allowed to subtract one from another. $\int_{\mathbb{R}^d} f^+(x)dx - \int_{\mathbb{R}^d} f^-(x)dx = \int_{\mathbb{R}^d} f^+(x+y_0)dx - \int_{\mathbb{R}^d} f^-(x+y_0)dx$. And this implies the desired conclusion.

□

51 (Example 9) It is enough for us to show that

$$\lim_{n\to\infty} f(x+n) = 0 \text{ a.e } x \in [0,1).$$

STEP 1. Let us consider $\int_{[0,1)} \sum_{n=0}^{\infty} |f(x+n)|dx$. By Theorem 4.6, we have

$$\int_{[0,1)} \sum_{n=1}^{\infty} |f(x+n)|dx = \sum_{n=0}^{\infty} \int_{[0,1)} |f(x+n)|dx.$$

By Theorem 4.13, we have

$$\sum_{n=0}^{\infty} \int_{[0,1)} |f(x+n)|dx = \sum_{n=0}^{\infty} \int_{[n,n+1)} f(x)dx.$$

By Theorem 4.11, we have

$$\sum_{n=0}^{\infty} \int_{[n,n+1)} |f(x)|dx = \int_{[0,\infty)} |f(x)|dx < \infty.$$

STEP 2. From the argument above, we find out that $\sum_{n=1}^{\infty} |f(x+n)|$ is integrable on $[0,1)$. So we have

$$\sum_{n=0}^{\infty} |f(x+n)| < \infty \text{ a.e } x \in [0,1).$$

For a fixed $x \in [0,1)$, if $\sum_{n=0}^{\infty} |f(x+n)|$ converges, $\lim_{n\to\infty} |f(x+n)| = 0$ according to knowledge of basic calculus. So $\lim_{n\to\infty} |f(x+n)| = 0$ a.e $x \in [0,1)$.

□

52 (Example 10) Let us recall that $E \in \mathscr{M}; E \subset \mathbb{R}; a \in \mathbb{R}\setminus\{0\}$ then $m^*(aE) = |a|m^*(E)$ and $aE \in \mathscr{M}$. (See Theorem 2.5)

STEP 1. (non-negative measurable simple function I) Let $f(x) \stackrel{\text{def}}{=} c\chi_E(x)$. Then $g(x) = f(ax) = c\chi_E(ax) = c\chi_{a^{-1}E}(x)$. $\int_I f(x)dx = cm(E\cap I)$. $\int_J g(x)dx = \int_J c\chi_{a^{-1}E}(x)dx = cm(a^{-1}E \cap J) = cm(a^{-1}(E\cap I)) = \frac{c}{|a|}m(E\cap I) = \frac{1}{|a|}\int_I f(x)dx$. So $\int_I f(x)dx = |a|\int_J g(x)dx$.

STEP 2. (non-negative measurable simple function II) When $f(x) = \sum_{i=1}^{p} c_i\chi_{E_i}(x)$, by repeating the similar argument, we have $\int_I f(x)dx = |a|\int_J g(x)dx$.

STEP 3. (non-negative measurable function) Let $f(x)$ be a non-negative measurable function. Let $g(x) = f(ax)$ We can find a sequence of non-negative measurable simple functions $\{f_n(x)\}_{n\geqq 1}$ s.t $f_n(x) \nearrow f(x)$. Let $g_n(x) = f_n(ax)$. Then $g_n(x) \nearrow f(ax) = g(x)$. By the previous result, we have $\int_I f_n(x)dx = |a|\int_J g_n(x)$. By monotone convergence theorem $\lim_{n\to\infty}\int_I f_n(x)dx = \int_I f(x)dx$ and $\lim_{n\to\infty}|a|\int_J g_n(x)dx = |a|\int_J g(x)dx$.

STEP 4. (general measurable function) $f(x) = f^+(x) - f^-(x)$. Let $g(x) = f(ax)$. Then $g^+(x) = \max\{0, g(x)\} = \max\{0, f(ax)\} = f^+(ax)$. Similarly $g^-(x) = f^-(ax)$. Since $\int_I f^+(x)dx = |a|\int_J g^+(x)$ and $\int_I f^-(x)dx = |a|\int_J g^-(x)dx$ and one of them is finite, so by subtracting one from another, we have the desired conclusion.

□

53 (Exercise 7) Let n be a natural number. Since the both sides are finite, so we can subtract one from another. So we have $\int_{[a,x]}(f(t)-g(t))dt = 0$ for all $x \in [a, a+n] \subset \mathbb{R}$. By Example 6, $f(x) - g(x) = 0$ a.e $x \in [a, a+n]$. So $m(\{x \in [a, a+n] \mid f(x) - g(x) \neq 0\}) = 0$. Since this holds for all $n \in \mathbb{N}$, $m\left(\bigcup_{n=1}^{\infty}\{x \in [a, a+n) \mid f(x) - g(x) \neq 0\}\right) = 0$ And we have $m(\{x \in [a,\infty) \mid f(x) - g(x) \neq 0\}) = 0$. This implies that $f(x) = g(x)$ a.e $x \in [a,\infty)$. □

54 (Exercise 8) $\phi(x) \stackrel{\text{def}}{=} \chi_{\{x\in\mathbb{R}|f(x)\geqq 0\}}$. Since $0 \leqq \phi(x) \leqq 1$, $\phi(x)$ is bounded. $\int_{\mathbb{R}} f(x)\phi(x) = \int_{\{x\in\mathbb{R}|f(x)\geqq 0\}} f(x)dx = 0$. By properties of integral in §4.1, we have $f(x) = 0$ a.e $x \in \{x \in \mathbb{R} \mid f(x) \geqq 0\}$. (i.e $m(\{x \in \mathbb{R} \mid f(x) > 0\}) = 0$)

Similarly, let $\phi(x) \stackrel{\text{def}}{=} -\chi_{\{x\in\mathbb{R}|f(x)\leqq 0\}}$. We have $f(x) = 0$ a.e $x \in \{x \in \mathbb{R} \mid f(x) \leqq 0\}$ (i.e $m(\{x \in \mathbb{R} \mid f(x) < 0\}) = 0$)

By merging these two results, we have $m(\{x \in \mathbb{R} \mid f(x) \neq 0\}) = 0$. This implies that $f(x) = 0$ a.e $x \in \mathbb{R}$. □

55 (Theorem 4.14 L.D.C.T) We apply Fatou's lemma (Theorem 4.8) to $\{2g(x) - |f_k(x) - f(x)|\}_{k\geqq 1}$, where $2g(x) - |f_k(x) - f(x)| \geqq 0$ for all $k \geqq 1$ a.e $x \in E$. Let us recall that we suppose that $f_k(x) \geqq 0$ in the assumption of Fatou's lemma. However, even if $f_k(x) \geqq 0$ for all $k \geqq 1$ a.e $x \in E$, the conclusion of Fatou's lemma still holds.

Let $N \stackrel{\text{def}}{=} \bigcup_{k=1}^{\infty}\{x \in E \mid f_k(x) < 0\}$ then $m(N) = 0$. Since if $x \in E \setminus N$ then $f_k(x) \geqq 0$ for all $k \geqq 1$, we have

$$\int_{E\setminus N} \liminf_{k\to\infty} f_k(x)dx \leqq \liminf_{k\to\infty} \int_{E\setminus N} f_k(x)dx.$$

The left hand side is equal to $\int_E \liminf_{k\to\infty} f_k(x)dx$ and the right hand side is equal to $\liminf_{k\to\infty} \int_E f_k(x)dx$ because N is a measure zero set. ($\int_E = \int_{E\setminus N} + \int_N = \int_E$.)

STEP 1. First, we prove that $\sup_{k\geqq 1} |f_k(x)| \leqq g(x)$ a.e $x \in E$. $|f_k(x)| \leqq g(x)$ a.e $x \in E$ for each $k \in \mathbb{N}$. Let $N_k \stackrel{\text{def}}{=} \{x \in E \mid |f_k(x)| > g(x)\}$. Let us recall that

$$\bigcup_{k=1}^{\infty} N_k = \{x \in E \mid \sup_{k\geqq 1} |f_k(x)| > g(x)\}.$$

Then $m(\bigcup_{k=1}^{\infty} N_k) = 0$. So $\sup_{k\geqq 1} |f_k(x)| \leqq g(x)$ a.e $x \in E$.

STEP 2. Next, we prove that $\sup_{k\geqq 1} |f_k(x) - f(x)| \leqq 2g(x)$ a.e $x \in E$. Since

- if $\lim_{k\to\infty} f_k(x)$ exists, then $\lim_{k\to\infty} |f_k(x)| \leqq \sup_{k\geqq 1} |f_k(x)|$,
- $\lim_{k\to\infty} |f_k(x)| = |f(x)|$ a.e $x \in E$,

we conclude that

$$|f(x)| \leqq g(x) \text{ a.e } x \in E.$$

By triangular inequality and the previous two results, we have

$$\begin{aligned} \sup_{k\geqq 1} |f_k(x) - f(x)| &\leqq \sup_{k\geqq 1} |f_k(x)| + |f(x)| \\ &\leqq g(x) + g(x) = 2g(x) \text{ a.e } x \in E. \end{aligned}$$

Equivalently,

$$2g(x) - \sup_{k\geqq 1} |f_k(x) - f(x)| \geqq 0 \text{ a.e } x \in E.$$

STEP 3. Note that

$$2g(x) - |f_k(x) - f(x)| \geqq 2g(x) - \sup_{k\geqq 1} |f_k(x) - f(x)|,$$

which explains that $2g(x) - |f_k(x) - f(x)| \geqq 0$ for all $k \geqq 1$ a.e $x \in E$. By Fatou's lemma, we have

$$\int_E \liminf_{k\to\infty}(2g(x) - |f_k(x) - f(x)|)dx \leqq \liminf_{k\to\infty} \int_E (2g(x) - |f_k(x) - f(x)|)dx$$

The left hand side is

$$\int_E \liminf_{k\to\infty}(2g(x) - |f_k(x) - f(x)|)dx = \int_E 2g(x)dx.$$

The right hand side is

$$\begin{aligned} \liminf_{k\to\infty} \int_E (2g(x) - |f_k(x) - f(x)|)dx &\stackrel{*1}{=} \liminf_{k\to\infty} \left(\int_E 2g(x)dx - \int_E |f_k(x) - f(x)|dx \right) \\ &\stackrel{*2}{\leqq} \int_E 2g(x)dx - \limsup_{k\to\infty} \int_E |f_k(x) - f(x)|dx \end{aligned}$$

- (∗1) $0 \leqq \int_E 2g(x) < \infty$. By Theorem 4.10 we can assure that linearity holds.
- (∗2) recall that $\liminf_{n\to\infty} -a_n = -\limsup_{n\to\infty} a_n$

Finally we have

$$\int_E 2g(x)dx \leqq \int_E 2g(x)dx - \limsup_{n\to\infty} \int_E |f_k(x) - f(x)|dx.$$

Since $\int_E 2g(x) < \infty$, we may subtract it from the both sides. And we have

$$\limsup_{n\to\infty} \int_E |f_k(x) - f(x)|dx = 0.$$

By triangular inequality, this also implies that

$$\limsup_{n\to\infty} \left| \int_E (f_k(x) - f(x))dx \right| \leqq \limsup_{n\to\infty} \int_E |f_k(x) - f(x)|dx = 0.$$

Since $f_k(x)$ is integrable, (again by Theorem 4.10),

$$\limsup_{n\to\infty} \left| \int_E f_k(x)dx - \int_E f(x)dx \right| = 0.$$

This implies the desired conclusion.

Note.

- $f_k(x) \geqq 0$ a.e $x \in E$ holds for each $k \geqq 1$.
- $f_k(x) \geqq 0$ for all $k \geqq 1$ a.e $x \in E$.

these two statements have the different meaning, but they are equivalent. (You can prove this like Step 1.) □

56 (Theorem 4.15) The Lebesgue Dominated Convergence Theorem holds even if the condition $f_k(x) \xrightarrow{\text{a.e}} f(x)$ changes to $f_k(x) \xrightarrow{m} f(x)$. Let us recall that

$$a_n \to a \in \mathbb{R}$$

if and only if

$$\forall \{n_k\}_{k\geqq 1} \subset \mathbb{N},\ \exists \{n_{k_\ell}\} \text{ s.t } a_{n_{k_\ell}} \to a,$$

where $\{n_{k_\ell}\}$ is a further subsequence of $\{n_k\}$.

Let us consider a sequence $a_n \overset{\text{def}}{=} \int_E |f_n(x) - f(x)|dx$. Let n_k be an arbitrary subsequence of natural numbers. We show that there exists a sub-subsequence n_{k_ℓ} s.t

$$\lim_{\ell\to\infty} \int_E |f_{n_{k_\ell}}(x) - f(x)|dx = 0.$$

This implies that $\lim_{n\to\infty} \int_E |f_n(x) - f(x)|dx = 0$.

STEP 1. Since $f_n(x) \xrightarrow{m} f(x)$, $\forall n_k$ (subsequence) there exists n_{k_ℓ} (subsubsequence) s.t $f_{n_{k_\ell}}(x) \xrightarrow{\text{a.u}} f(x)$ (Theorem 3.17). $\xrightarrow{\text{a.u}}$ always implies $\xrightarrow{\text{a.e}}$. So there exists $f_{n_{k_\ell}}(x) \xrightarrow{\text{a.e}} f(x)$.

STEP 2. Obviously, $\sup_{\ell \geqq 1} |f_{n_{k_\ell}}(x)| \leqq \sup_{n \geqq 1} |f_n(x)| \leqq g(x) \in L(E)$ a.e $x \in E$, so by Theorem 4.14, we have $\lim_{\ell\to\infty} \int_E |f_{n_{k_\ell}}(x) - f(x)| dx = 0$. So we have $\lim_{n\to\infty} \int_E |f_n(x) - f(x)| dx = 0$.

STEP 3. By triangular inequality, $\lim_{n\to\infty} |\int_E (f_n(x) - f(x)) dx| = 0$. Since $\int_E f_n(x) dx$ is finite, linearlity holds in integral. So $\lim_{n\to\infty} |\int_E f_n(x)dx - \int_E f(x)dx| = 0$. This implies the desired conclusion.

□

57 (Example 12) All we have to do is prove that

$$\lim_{n\to\infty} \frac{\int_{[0,1]} \frac{x \sin x}{1+(nx)^\alpha} dx}{\frac{1}{n}} = \lim_{n\to\infty} \int_{[0,1]} \frac{(nx) \sin x dx}{1+(nx)^\alpha} dx = 0.$$

STEP 1. Let $f_n(x) \overset{\text{def}}{=} \frac{(nx)\sin x}{1+(nx)^\alpha}$. Then $|f_n(x)| \leqq \frac{nx}{1+(nx)^\alpha}$. We hope to find an integrable bound function. Let $g_n(x) \overset{\text{def}}{=} \frac{nx}{1+(nx)^\alpha}$. $g_n'(x) = \frac{n - n(\alpha-1)(nx)^\alpha}{1+(nx)^\alpha}$. When $(nx)^\alpha = \frac{1}{\alpha-1}$ (i.e $x = x_n \overset{\text{def}}{=} \frac{1}{n}(\frac{1}{\alpha-1})^{\frac{1}{\alpha}}$), $g_n'(x) = 0$. (There exists $N_\alpha \in \mathbb{N}$ s.t $\forall n > N_\alpha$ $x_n \in (0,1)$.) Then $g_n(x)$ takes the maximum value $M_\alpha = \frac{(\frac{1}{\alpha-1})^{\frac{1}{\alpha}}}{1+\frac{1}{\alpha-1}}$ which is not related to n. So $|f_n(x)| \leqq M_\alpha \in L([0,1])$, $\forall n > N_\alpha$. (We may ignore $n = 1, 2 \cdots N_\alpha$ because we take $\lim_{n\to\infty} \cdot$)

STEP 2. By Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_{[0,1]} f_n(x) dx = \int_{[0,1]} \lim_{n\to\infty} f_n(x) dx = \int_{[0,1]} 0 \ dx = 0.$$

□

58 (Example 13) Our goal is to prove that

$$\lim_{n\to\infty} \frac{\int_{[\alpha,\infty)} \frac{x \exp(-n^2x^2)}{1+x^2} dx}{\frac{1}{n^2}} = 0.$$

So we prove that

$$\lim_{n\to\infty} \int_{[\alpha,\infty)} \frac{n^2 x \exp(-n^2x^2)}{1+x^2} dx = 0.$$

By Example 10, we have ($t = nx$)

$$= \lim_{n\to\infty} \int_{[n\alpha,\infty)} \frac{t \exp(-t^2)}{n^2+t^2} dt$$

And

$$= \lim_{n\to\infty} \int_{[0,\infty)} \frac{t \exp(-t^2)}{n^2+t^2} \chi_{[n\alpha,\infty)}(t) dt.$$

Since

$$\frac{t\exp(-t^2)}{n^2+t^2}\chi_{[n\alpha,\infty)}(t) \leqq \frac{t\exp(-t^2)}{1+t^2} \leqq \exp(-t^2) \in L([0,\infty)),$$

we can apply Lebesgue Dominated Convergence Theorem. $\lim_{n\to\infty} \frac{t\exp(-t^2)}{n^2+t^2}\chi_{[n\alpha,\infty)}(t) = 0$. So the proof is complete. (Notice) $\exp(-t^2) \leqq \sum_{n=0}^{\infty} 2^{-n}\chi_{[n,n+1)}(t) \in L([0,\infty))$. □

59 (Exercise 1) We show that

$$\int_a^x f(t)\phi(t)dt = \phi(x) - \phi(a), \forall x \in [a,b]$$

By assumption, we have $\int_a^x f(t)\phi_n(t)dt = \phi_n(x) - \phi_n(a)$. By taking limit, we have $\lim_{n\to\infty}\int_a^x f(t)\phi_n(t)dt = \lim_{n\to\infty}(\phi_n(x) - \phi_n(a))$. By Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_a^x f(t)\phi_n(t)dt = \int_a^x f(t)\phi(t)dt$$

because $|f(t)\phi_n(t)| \leqq F(t) \in L([a,x])$. ($F(t) \in L([a,b])$ implies that $F(t) \in L([a,x])$ for all $x \in [a,b]$) The right hand side is $\phi(x) - \phi(a)$. So the proof is complete. □

60 (Exercise 2) We use Lebesgue Dominated Convergence Theorem. (converge in measure version) Suppose that $\cos(nx) \overset{m}{\to} 0$. $\cos(nx) \overset{m}{\to} 0$ on $[-\pi,\pi)$ if and only if $\cos^2(nx) \overset{m}{\to} 0$ on $[-\pi,\pi)$. $|cos^2(nx)| \leqq 1 \in L([-\pi,\pi))$. By Lebesgue Dominated Convergence Theorem (converge in measure version), we have $\lim_{n\to\infty}\int_{[-\pi,\pi)}\cos^2(nx)dx = \int_{[-\pi,\pi)} 0\ dx = 0$. However this conclusion is false because

$$\int_{[-\pi,\pi)}\cos^2(nx)dx = \int_{[-\pi,\pi)}\frac{\cos(2nx)+1}{2}dx = \pi \neq 0.$$

□

61 (Exercise 3) Since $|g(x)| \leqq \int_{(0,\infty)}\frac{|f(t)|}{x+t}dt \leqq \int_{(0,\infty)}\frac{f(t)}{x}dx < \infty$, $g(x+h)-g(x)$ is well-defined. (i.e not $\infty-\infty$. both $g(x+h), g(x)$ are finite.) We show that $\lim_{h\to 0}|g(x+h)-g(x)| = 0$ for all $x \in (0,\infty)$. Let $\{h_n\}_{n\geqq 1}$ be a sequence of real numbers with $h_n \to 0$ as $n \to \infty$. And we show that $\lim_{n\to\infty}|g(x+h_n)-g(x)| = 0$.

Since $|h_n| \to 0$, we may assume that $|h_n| \leqq \frac{x}{2}$ with out loss of generality. Then $\frac{x}{2}+t \leqq x+t+h_n$. Note that $0 < \frac{x^2}{2} < (\frac{x}{2}+t)(x+t) \leqq (x+t+h_n)(x+t)$. So we have

$$\frac{|h_n|\cdot|f(t)|}{(x+t+h_n)(x+t)} \leqq \frac{x}{2}\cdot\frac{2}{x^2}\cdot|f(t)| = \frac{|f(t)|}{x} \in L((0,\infty)).$$

Finally,

$$\begin{aligned}\lim_{n\to\infty}|g(x+h_n)-g(x)| &\overset{*1}{=} \lim_{n\to\infty}\left|\int_{(0,\infty)}\frac{-h_nf(t)}{(x+t+h_n)(x+t)}dt\right| \\ &\overset{*2}{\leqq} \lim_{n\to\infty}\int_{(0,\infty)}\frac{|h_n|\cdot|f(t)|}{(x+t+h_n)(x+t)}dt \\ &\overset{*3}{=} \int_{(0,\infty)} 0\ dt = 0.\end{aligned}$$

Now the proof is complete.

- ($*1$) Integral has linearity $\int f_1 dx + \int f_2 dx = \int (f_1 + f_2) dx$ when at least one of them is integrable.
- ($*2$) triangular inequality
- ($*3$) L.D.C.T.

□

62 (Exercise 4) We can answer this question without employing Lebesgue's Dominated Convergence Theorem. However, we present a solution with L.D.C.T. $\int_{E_k} |f(x)|dx = \int_E |f(x)|\chi_{E_k}(x)dx$. Since $|f(x)\chi_{E_k}| \leqq |f(x)| \in L(E)$, by L.D.C.T

$$\lim_{k\to\infty} \int_E |f(x)|\chi_{E_k}(x)dx = \int_E \lim_{k\to\infty} |f(x)|\chi_{E_k}(x)dx \overset{*}{=} \int_E 0\ dx.$$

- ($*$) Fix $x \in E$. For all $|f(x)| > 0$, if k is sufficiently large $|f(x)| \geqq \frac{1}{k}$. Then $\chi_{E_k}(x) = 0$.

□

63 (Exercise 5) Let us recall that a sequence $\{a_n\}_{n\geqq 1}$ converges to a, (i.e. $a_n \to a$) if and only if $\forall n_k$ (subsequene) there exists n_{k_m} (subsubsequence) s.t $a_{n_{k_m}} \to a$. Let n_k be an arbitrary subsequence of natural numbers. We show that there exists n_{k_m} s.t $\int_E |f_{n_{k_m}}(x)g_{n_{k_m}}(x) - f(x)g(x)|$.

STEP 1.

$$\begin{aligned}
\frac{1}{\epsilon} \cdot \int_E |f_n(x) - f(x)|dx &\geqq \int_{\{x\in E||f_n(x)-f(x)|>\epsilon\}} \frac{1}{\epsilon}|f_n(x) - f(x)|dx \\
&\geqq \int_{\{x\in E||f_n(x)-f(x)|>\epsilon\}} 1\ dx \\
&= m(\{x \in E \mid |f_n(x) - f(x)| > \epsilon\})
\end{aligned}$$

By taking $n \to \infty$, we have $f_n(x) \overset{m}{\longrightarrow} f(x)$ on E. So for all subsequence n_k there exists n_{k_m} s.t $f_{n_{k_m}}(x) \overset{\text{a.u}}{\longrightarrow} f(x)$. $\overset{\text{a.u}}{\longrightarrow}$ implies that $\overset{\text{a.e}}{\longrightarrow}$. So there exists $f_{n_{k_m}}(x) \overset{\text{a.e}}{\longrightarrow} f(x)$ on E.

STEP 2.

$$\begin{aligned}
&\int_E |f_{n_{k_m}}(x)g_{n_{k_m}}(x) - f(x)g(x)|dx \\
=& \int_E |f_{n_{k_m}}(x)g_{n_{k_m}}(x) - f_{n_{k_m}}(x)g(x) + f_{n_{k_m}}(x)g(x) - f(x)g(x)|dx \\
\leqq& \int_E |f_{n_{k_m}}(x)g_{n_{k_m}}(x) - f_{n_{k_m}}(x)g(x)|dx \\
+& \int_E |f_{n_{k_m}}(x)g(x) - f(x)g(x)|dx \\
=& \int_E |f_{n_{k_m}}(x)| \cdot |g_{n_{k_m}}(x) - g(x)|dx + \int_E |f_{n_{k_m}}(x) - f(x)| \cdot |g(x)|dx \\
\overset{*}{\leqq}& \int_E M \cdot |g_{n_{k_m}}(x) - g(x)|dx + \int_E |f_{n_{k_m}}(x) - f(x)| \cdot |g(x)|dx
\end{aligned}$$

- $(*)\ \sup_{m\geqq 1}|f_{n_{k_m}}(x)| \leqq \sup_{n\geqq 1}|f_n(x)| \leqq M$.

In the last part of the inequality above, $|f_{n_{k_m}}(x) - f(x)|\cdot|g(x)| \leqq 2M\cdot|g(x)| \in L(E)$, we can apply Lebesgue Dominated Convergence Theorem. ($\lim_{m\to\infty}|f_{n_{k_m}}(x)| \leqq \sup_{m\geqq 1}|f_{n_{k_m}}(x)|$). By taking $m \nearrow \infty$, we have the desired conclusion.

□

64 (Exercise 6) We show that

$$\lim_{k\to\infty}\int_E |f_k(x) - f(x)|dx = 0.$$

Note that

$$\begin{aligned}\int_E |f_k(x) - f(x)|dx &\leqq \int_E \sup_{a\in E}|f_k(a) - f(a)|dx \\ &= m(E)\cdot\sup_{a\in E}|f_k(a) - f(a)|.\ (m(E) < \infty).\end{aligned}$$

Since $f_k(x) \xrightarrow{u} f(x)$,

$$\lim_{k\to\infty}\sup_{x\in E}|f_k(x) - f(x)| = 0.$$

Now the proof is complete. □

65 (Corollary 4.16) By Theorem 4.6,

$$\sum_{k=1}^{\infty}\int_E |f_k(x)|dx = \int_E \sum_{k=1}^{\infty}|f_k(x)|dx < \infty.$$

This implies that $\sum_{k=1}^{\infty}|f_k(x)| < \infty$ a.e $x \in E$. Let $S_n(x) \stackrel{\text{def}}{=} \sum_{k=1}^{n} f_k(x)$ and $\lim_{n\to\infty} S_n(x)$ exists a.e $x \in E$. ($\because$ absolute convergence) Let

$$S(x) \stackrel{\text{def}}{=} \begin{cases}\lim_{n\to\infty} S_n(x) & \text{if the limit exists} \\ 0 & \text{otherwise}\end{cases}.$$

$S(x)$ is a measurable function and $\lim_{n\to\infty} S_n(x) = S(x)$ a.e $x \in E$. $\sup_{n\geqq 1}|S_n(x)| \leqq \sum_{k=1}^{\infty}|f_k(x)| \in L(E)$. By Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_E S_n(x)dx = \int_E S(x)dx$$

The left hand side is

$$\lim_{n\to\infty}\int_E S_n(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n}\int_E f_k(x)dx.$$

So now we have the desired conclusion. □

66 (Theorem 4.17)

STEP 1. Let $\{h_n\}_{n\geqq 1} \subset \mathbb{R}$ be a sequence with $h_n \to 0$. By the definition of differentiation,

$$\frac{\partial}{\partial y}\int_E f(x,y)dx = \lim_{n\to\infty}\frac{1}{h_n}\left(\int_E f(x,y+h_n)dx - \int_E f(x,y)dx\right).$$

Since $f(x,y)$ is integrable with respect to x for all $y \in (a,b)$,

$$\frac{1}{h_n}\left(\int_E f(x,y+h_n)dx - \int_E f(x,y)dx\right),$$

is well-defined. ($\infty - \infty$ does not happen.) Since integral has linearity,

$$\frac{1}{h_n}\left(\int_E f(x,y+h_n)dx - \int_E f(x,y)dx\right) = \int_E \frac{f(x,y+h_n)-f(x,y)}{h_n}dx.$$

STEP 2. Since $f(x,y)$ is differentiable with respect to $y \in (a,b)$, there exists $c_n \in (y, y+h_n)$ or $c_n \in (y+h_n, y)$

$$\frac{f(x,y+h_n)-f(x,y)}{h_n} = \frac{\partial}{\partial y}f(x,y)\Big|_{y=c_n}.$$

by mean-value theorem. By assumption, $\left|\frac{\partial f(x,y)}{\partial y}\right| \leqq F(x) \in L(E)$, so we have

$$\sup_{n\geqq 1}\left|\frac{f(x,y+h_n)-f(x,y)}{h_n}\right| = \sup_{n\geqq 1}\left|\frac{\partial}{\partial y}f(x,y)\Big|_{y=c_n}\right| \leqq F(x),\ \forall n \in \mathbb{N}.$$

$F(x)$ is not related to n. By Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}\lim_{n\to\infty}\int_E \frac{f(x,y+h_n)-f(x,y)}{h_n}dx &= \int_E \lim_{n\to\infty}\frac{f(x,y+h_n)-f(x,y)}{h_n}dx \\ &= \int_E \frac{\partial}{\partial y}f(x,y)dx.\end{aligned}$$

Now we have the desired conclusion.

□

67 (Example 14) □

68 (Exercise 7) Suppose that $\int_E f(x)\cos x dx = 1$, and let us try to derive a contradiction. Note that

$$\int_E f(x)dx - \int_E f(x)\cos x dx = 0,$$

hence

$$\int_E f(x)(1-\cos x)dx = 0,$$

because $|\int_E f(x)dx| < \infty$ and thus linearity holds in integral. (Theorem 4.10)

Since $f(x) \geqq 0$, $1-\cos x \geqq 0$, $f(x)(1-\cos x) \geqq 0$. By properties about integral of non-negative measurable functions, $\int_E f(x)(1-\cos x)dx = 0$ implies that $f(x)(1-\cos x) = 0$ a.e $x \in E$. Therefore $f(x) = 0$ a.e $x \in E$ or $1-\cos x = 0$ a.e $x \in E$ holds.

case 1. ($f(x) = 0$ a.e $x \in E$) Suppose $f(x) = 0$ a.e $x \in E$ then $\int_E f(x)dx = 0$. (contradiction!!)

case 2. ($1 - \cos x = 0$ a.e $x \in E$) Suppose that $1 - \cos x = 0$ a.e $x \in E$. However $\{x \in E \mid 1 - \cos x = 0\} \subset \bigcup_{n\in\mathbb{Z}}\{2n\pi\}$ and $m(\bigcup_{n\in\mathbb{Z}}^{\infty}\{2n\pi\}) = 0$. So $1 - \cos x = 0$ a.e $x \in E$ can not occur except $m(E) = 0$. However if $m(E) = 0$, then $\int_E f(x)dx = 0 \neq 1$.

The both cases above contradict to the assumption. So we conclude that

$$\int_E f(x)\cos x dx \neq 1.$$

□

69 (Exercise 8) First, $\sum_{n=1}^{\infty}\int_{\mathbb{R}}|f_n(x) - f(x)|dx \leqq \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} < \infty$. By Theorem 4.6, $\int_{\mathbb{R}}\sum_{n=1}^{\infty}|f_n(x) - f(x)|dx < \infty$. By properties of integral, this implies that $\sum_{n=1}^{\infty}|f_n(x) - f(x)| < \infty$ a.e $x \in \mathbb{R}$. So $\lim_{n\to\infty}|f_n(x) - f(x)| = 0$ a.e $x \in \mathbb{R}$. (See books of basic calculus.) □

70 (Exercise 9) Let us consider

$$\begin{aligned}
\int_{[2,\infty)}\sum_{n=2}^{\infty}\left|a_n n^{-x}\right| dx &\overset{*1}{=} \sum_{n=2}^{\infty}\int_{[2,\infty)}\left|a_n n^{-x}\right| \\
&\overset{*2}{=} \sum_{n=2}^{\infty}|a_n|\int_{[2,\infty)} n^{-x} \\
&\overset{*3}{=} \sum_{n=2}^{\infty}|a_n|\frac{1}{n^2\log n} \\
&\overset{*4}{\leqq} \sum_{n=2}^{\infty}\frac{1}{n^2} < \infty.
\end{aligned}$$

- $(*1)$ Theorem 4.6
- $(*2)$ linearity of integral
- $(*3, 4)$ by assumption

By Corollary 4.16, we have

$$\int_{[2,\infty)}\sum_{n=2}^{\infty}a_n n^{-x}dx = \sum_{n=2}^{\infty}\int_{[2,\infty)}a_n n^{-x}dx$$

and the right hand side is $\sum_{n=2}^{\infty}\frac{a_n}{n^2\log n}$.

□

71 (Exercise 10) Let $\{h_n\}_{n\geqq 1}$ be a sequence with $h_n \to 0$. $F(y + h_n) - F(y) = \int_E f(x, y + h_n)dx - \int_E f(x, y)dx$. Since $|f(x, y)| \leqq g(x) \in L(E)$ for all $y \in \mathbb{R}^d$, both $\int_E f(x, y + h_n)dx$, $\int_E f(x, y)dx$ are finite, hence well-defined. (not $\infty - \infty$) By linearity,

$\int_E f(x,y+h_n)dx - \int_E f(x,y)dx = \int_E (f(x,y+h_n) - f(x,y))$. And $|f(x,y+h_n)-f(x,y)| \leqq 2g(x) \in L(E)$. By Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_E (f(x,y+h_n)-f(x,y))dx = \int_E \lim_{n\to\infty}(f(x,y+h_n)-f(x,y))dx \overset{*}{=} \int_E 0\ dx = 0.$$

- $(*)$ holds because $f(x,y)$ is continuous with respect to $y \in \mathbb{R}^d$.

□

§ 4.3

72 (Theorem 4.18)

STEP 1. We have already shown that there exists a sequence of Lebesgue measurable simple functions defined on E $\{f_n(x)\}_{n\geqq 1}$ with a compact support s.t $|f_n(x)| \leqq |f(x)|$ and $f_n(x) \to f(x)$ as $n \to \infty$. Since $|f_n(x) - f(x)| \leqq |f_n(x)| + |f(x)| \leqq 2|f(x)| \in L(E)$, by applying Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int_E |f_n(x)-f(x)|dx = \int_E \lim_{n\to\infty}|f_n(x)-f(x)|dx = \int_E 0\ dx.$$

This implies that for an arbitrary positive number $\epsilon > 0$, there exists sufficiently large n_0 s.t $\int_E |f_{n_0}(x) - f(x)|dx < \frac{\epsilon}{2}$. Let $\tilde{f}(x) \overset{\text{def}}{=} f_{n_0}(x)$.

STEP 2. Since $\tilde{f}(x)$ is a measurable simple function, we suppose that $\tilde{f}(x) = \sum_{i=1}^p a_i\chi_{E_i}(x)$ where $\{a_i\}_{i=1}^p \subset \mathbb{R}$, $E_i \in \mathscr{M}$, $E = \bigcup_{i=1}^p E_i$. Let $M \overset{\text{def}}{=} \max\{|a_i|\}_{i=1}^p$. Then $|\tilde{f}(x)| \leqq M < \infty$.

$\tilde{f}(x)$ has a compact support, so we may suppose that if $a_i \neq 0$, $E_i \subset B$: a bounded ball on $\mathbb{R}^d$. We may regard $\tilde{f}(x)$ as a measurable function defined on B because $\tilde{f}(x) = \sum_{i=1,a_i\neq 0}^p a_i\chi_{E_i}(x) + 0\cdot\chi_{B\setminus\bigcup_{i=1,a_i\neq 0}^p E_i}(x)$.

Now we apply Corollary 3.19 to $\tilde{f}(x)$ as a measurable function defined on B. We have $g(x) \in C(\mathbb{R}^d)$ s.t $m(\{x \in B \mid \tilde{f}(x) \neq g(x)\}) < \delta = \frac{\epsilon}{4M}$. Since $|\tilde{f}(x)| \leqq M$, $|g(x)| \leqq M$ on $\mathbb{R}^d$. Moreover, $g(x)$ has a compact support. (if $x \notin B$, $g(x) = 0$) (See Corollary 3.19.)

$$\begin{aligned}
\int_E |\tilde{f}(x)-g(x)|dx &= \int_{E\cap B}|\tilde{f}(x)-g(x)|dx + \int_{E\setminus B}|\tilde{f}(x)-g(x)|dx \\
&\overset{*1}{=} \int_{E\cap B}|\tilde{f}(x)-g(x)|dx \\
&\overset{*2}{\leqq} \int_B |\tilde{f}(x)-g(x)|dx \\
&= \int_{\{x\in B\mid \tilde{f}\neq g\}}|\tilde{f}(x)-g(x)|dx \\
&\overset{*3}{\leqq} \int_{\{x\in B\mid \tilde{f}\neq g\}} 2M dx = 2M\cdot m(\{x\in B \mid \tilde{f}\neq g\}) < \frac{\epsilon}{2}
\end{aligned}$$

- (∗1) $x \notin B$, $\tilde{f}(x), g(x) = 0$.
- (∗2) $E \cap B \subset B$
- (∗3) $|\tilde{f}(x) - g(x)| \leqq |\tilde{f}(x)| + |g(x)| \leqq 2M$

STEP 3.

$$\int_E |f(x) - g(x)|dx \leqq \int_E |f(x) - \tilde{f}(x)|dx + \int_E |\tilde{f}(x) - g(x)|dx < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

□

73 (Corollary 4.19, 4.20) We may find $\{g_k(x)\} \subset C(\mathbb{R}^d)$ s.t $\int_E |f(x) - g_k(x)|dx < \frac{1}{k^2}$. Then $\sum_{k=1}^{\infty} \int_E |f(x) - g_k(x)|dx = \int_E \sum_{k=1}^{\infty} |f(x) - g_k(x)| < \infty$. $\sum_{k=1}^{\infty} |f(x) - g_k(x)| \in L(E)$ hence $\sum_{k=1}^{\infty} |f(x) - g_k(x)| < \infty$ a.e $x \in E$. So $\lim_{k\to\infty} |f(x) - g_k(x)| = 0$ a.e $x \in E$.

We present an alternative solution. We may find $\{g_k(x)\} \subset C(\mathbb{R}^d)$ s.t $\int_E |f(x) - g_k(x)|dx < \frac{1}{k}$. Then $g_k(x) \xrightarrow{m} f(x)$ on E because

$$\begin{aligned}
\epsilon \cdot m(\{x \in E \mid |f(x) - g_k(x)| > \epsilon\}) &= \int_{\{x\in E||f(x)-g_k(x)|>\epsilon\}} \epsilon \, dx \\
&\leqq \int_{\{x\in E||f(x)-g_k(x)|>\epsilon\}} |f(x) - g_k(x)|dx \\
&\leqq \int_E |f(x) - g_k(x)|dx \to 0
\end{aligned}$$

Since $g_k(x) \xrightarrow{m} f(x)$, for every subsequence k_ℓ (we may let $k_\ell = \ell$ here), we can find a subsubsequence k_{ℓ_m} s.t $g_{k_{\ell_m}}(x) \xrightarrow{\text{a.u}} f(x)$ on E. (hence $\xrightarrow{\text{a.e}} f(x)$ on E). So the subsubsequence is the desired sequence. □

74 (Example 1) Suppose that $f(x) = 0$ a.e $x \in \mathbb{R}^d$ is not true. In other words, suppose that $m(\{x \in \mathbb{R}^d \mid f(x) > 0\}) > 0$ or $m(\{x \in \mathbb{R}^d \mid f(x) < 0\}) > 0$. Without loss of generality, we may suppose that $m(\{x \in \mathbb{R}^d \mid f(x) > 0\}) > 0$.

Let $\tilde{E} \overset{\text{def}}{=} \{x \in \mathbb{R}^d \mid f(x) > 0\}$. We can find a bounded measurable subset of $\tilde{E}$, E with $m(E) > 0$. Let $E_k \overset{\text{def}}{=} \tilde{E} \cap B(0, k)$. Then $E_k \nearrow \tilde{E}$ and $m(E_k) \nearrow m(\tilde{E}) > 0$. Therefore we can find $k_0 \in \mathbb{N}$ s.t $m(E_{k_0}) > 0$. Let $E \overset{\text{def}}{=} E_{k_0}$.

We apply Corollary 4.19, 4.20 to $\chi_E(x)$. ($\chi_E(x) \in L(\mathbb{R}^d)$.) We can find a sequence of continuous functions $\{g_k(x)\}_{k\geqq 1} \subset C(\mathbb{R}^d)$ with a bounded support s.t $\int_{\mathbb{R}^d} |\chi_E(x) - f_k(x)|dx \to 0$ and $g_k(x) \xrightarrow{\text{a.e}} \chi_E(x)$ on $\mathbb{R}^d$. Let us pay attention to the fact that $|g_k(x)| \geqq 1$ because $|\chi_E(x)| \leqq 1$. (In Theorem 4.18 or Corollary 3.19, $|f(x)| \leqq M \Rightarrow |g(x)| \leqq M$)

Since $|f(x)g_k(x)| \leqq |f(x)| \in L(\mathbb{R}^d)$, by Lebesgue Dominated Convergence theorem, we have

$$\begin{aligned}
\lim_{k\to\infty} \int_{\mathbb{R}^d} f(x)g_k(x)dx &= \int_{\mathbb{R}^d} \lim_{k\to\infty} f(x)g_k(x)dx \\
&= \int_{\mathbb{R}^d} f(x)\chi_E(x)dx \\
&= \int_E f(x)dx \overset{*}{>} 0
\end{aligned}$$

- $(*)$ $f(x) > 0$ on E and $m(E) > 0$ then $\int_E f(x)dx > 0$.

However, $\int_{\mathbb{R}^d} f(x)g_k(x)dx = 0$ by assumption. (contradiction!!) So $m(\{x \in \mathbb{R}^d \mid f(x) > 0\}) = 0$. Similarly $m(\{x \in \mathbb{R}^d \mid f(x) < 0\}) = 0$. Now the proof is complete. □

75 (Theorem 4.21) Let $\epsilon > 0$ be an arbitrary positive number.

STEP 1. By Theorem 4.18, we can find a continuous function $g \in C(\mathbb{R}^d)$ with a bounded support s.t

$$\int_{\mathbb{R}^d} |f(x) - g(x)|dx < \frac{\epsilon}{4}.$$

Let $h(x) \stackrel{\text{def}}{=} f(x) - g(x)$. Then $\int_{\mathbb{R}^d} |h(x)|dx < \frac{\epsilon}{4}$.

STEP 2. Suppose that $\text{supp}(g) \subset K \stackrel{\text{def}}{=} \overline{B}(0, M), (0 < M < \infty)$. Since we take $x_0 \to 0$, we may consider that $|x_0| \leqq 1$. Therefore, $K_1 \stackrel{\text{def}}{=} \overline{B}(0, M+1)$ contains the support of $g(x + x_0) - g(x)$. And we have

$$\int_{\mathbb{R}^d} |g(x + x_0) - g(x)|dx = \int_{K_1} |g(x + x_0) - g(x)|dx.$$

Let $K_2 \stackrel{\text{def}}{=} \overline{B}(0, M+2)$. $g(x)$ is continuous on $\mathbb{R}^d$, so is on K_2 which is a bounded closed set. Let us recall that a continuous function defined on a bounded closed (compact) set is uniformly continuous. Therefore $\exists \delta > 0$, $\forall x, y \in K_2$ with $|x - y| < \delta$, $|g(x) - g(y)| < \frac{\epsilon}{2m(K_1)}$. If $|x_0| < \delta$, we have $\forall x \in K_1$, $|g(x + x_0) - g(x)| < \frac{\epsilon}{2m(K_1)}$. So we have

$$\begin{aligned}\int_{\mathbb{R}^d} |g(x + x_0) - g(x)|dx &= \int_{K_1} |g(x + x_0) - g(x)|dx \\ &\leqq \int_{K_1} \frac{\epsilon}{2m(K_1)}dx = \frac{\epsilon}{2}\end{aligned}$$

STEP 3.

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x + x_0) - f(x)|dx &\leqq \int_{\mathbb{R}^d} |g(x + x_0) - g(x)|dx + \int_{\mathbb{R}^d} |h(x + x_0) - h(x)|dx \\ &\stackrel{*1}{\leqq} \frac{\epsilon}{2} + \int_{\mathbb{R}^d} |h(x + x_0)|dx + \int_{\mathbb{R}^d} |h(x)|dx \\ &\stackrel{*2}{\leqq} \frac{\epsilon}{2} + 2\int_{\mathbb{R}^d} |h(x)|dx \\ &\stackrel{*3}{<} \frac{\epsilon}{2} + \frac{\epsilon}{2}\end{aligned}$$

- $(*1)$ Step2 and triangular inequality
- $(*2)$ Theorem 4.13 states that translation does not change the value of integral on $\mathbb{R}^d$.
- $(*3)$ Step1

□

76 (Example 3)

STEP 1. $m(E) = \int_{\mathbb{R}^d} \chi_E(x)dx = \int_{\mathbb{R}^d} (\chi_E(x))^2 dx.$

STEP 2. $m(E \cap E_{+h}) = \int_{\mathbb{R}^d} \chi_{E \cap E_{+h}}(x)dx = \int_{\mathbb{R}^d} \chi_E(x) \cdot \chi_{E_{+h}}(x)dx.$

STEP 3.

$$\begin{aligned}
|m(E) - m(E \cap E_{+h})| &= \left| \int_{\mathbb{R}^d} (\chi_E(x))^2 dx - \int_{\mathbb{R}^d} \chi_E(x) \cdot \chi_{E_{+h}}(x)dx \right| \\
&\overset{*1}{\leqq} \int_{\mathbb{R}^d} \chi_E(x) \left| \chi_E(x) - \chi_{E_{+h}}(x) \right| dx \\
&\overset{*2}{\leqq} \int_{\mathbb{R}^d} \left| \chi_E(x) - \chi_{E_{+h}}(x) \right| dx \\
&\overset{*3}{\leqq} \int_{\mathbb{R}^d} |\chi_E(x) - \chi_E(x-h)| \, dx \\
&\overset{*4}{\to} 0
\end{aligned}$$

- $(*1)$ triangular inequality
- $(*2)$ $\chi_E(x) \leqq 1$
- $(*3)$ $x \in E_{+h}$ if and only if $x - h \in E$
- $(*4)$ Theorem 4.21.

□

77 (Corollary 4.22) It is enough for us to prove that for all $\epsilon > 0$, there exists a step function with a compact support (a bounded support) s.t

$$\int_E |f(x) - \phi(x)| dx < \epsilon.$$

STEP 1. We have already proven that there exists a continuous function $g \in C(\mathbb{R}^d)$ with a compact support s.t

$$\int_E |f(x) - g(x)| dx < \frac{\epsilon}{2}.$$

So we prove that there exists a step function $\phi(x)$ with a compact support s.t

$$\int_E |g(x) - \phi(x)| dx < \frac{\epsilon}{2},$$

then the proof is complete.

STEP 2. Suppose that $\mathrm{supp}(g) \subset \prod_{i=1}^{d}(-N, N]$ where $N \in \mathbb{N}$. Let $I = \prod_{i=1}^{d}(-N, N]$. (This is a half open rectangle in $\mathbb{R}^d$). We define $I_{n,k} \overset{\text{def}}{=} \prod_{i=1}^{d}\left(\frac{k_i}{2^n}, \frac{k_i+1}{2^n}\right]$ where $n \in \mathbb{N}, k \in \mathbb{Z}^d$. Let

$$g_n(x) \overset{\text{def}}{=} \sum_{k \in \{-N\cdot 2^n, -N\cdot 2^n+1, \cdots, N\cdot 2^n-1\}^d} \inf_{a \in I_{n,k}} \{g(a)\} \cdot \chi_{I_{n,k}}(x).$$

This definition seems a bit complicated but we just divide I into small rectangles $\{I_{n,k}\}_k$ and take infimum of $g(x)$ in each rectangle. When n goes to infinity, the division of I becomes finer. Since $g(x)$ is continuous, $g_n(x) \nearrow g(x)$ as $n \to \infty$. $g_n(x) \leqq g_{n+1}(x)$ holds because for all $x_0 \in I_{n,k}$ we can find k' s.t $x_0 \in I_{n+1,k'} \subset I_{n,k}$ and $\inf A \geqq \inf B$ if $A \subset B$.

We apply monotone convergence theorem to $g_n(x)$ and we have $\lim_{n\to\infty} \int_E g_n(x)dx = \int_E g(x)dx$. ($g_n(x)$ is not necessarily non-negative, but we can consider the sequence of $\{g_n(x) - g_1(x)\}$. $g_1(x) \in L(E)$. See Example 3 in §4.1.)

$$\begin{aligned} 0 \leqq \int_E |g_n(x) - g(x)|dx &= \int_E (g(x) - g_n(x))dx \\ &\overset{*}{=} \int_{E\cap I} (g(x) - g_n(x))dx \\ &\leqq \int_I (g(x) - g_n(x))dx \\ &= \int_I g(x)dx - \int_I g_n(x)dx < \frac{\epsilon}{2} \end{aligned}$$

for sufficiently large $n_0 \in \mathbb{N}$. Let $\phi(x) \overset{\text{def}}{=} g_{n_0}(x)$.

- $(*)$ $x \notin I$, $g(x), g_n(x) = 0$.

STEP 3. Now the proof is almost complete. $\int_E |f(x) - \phi(x)|dx \leqq \int_E |f(x) - g(x)|dx + \int_E |g(x) - \phi(x)|dx < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. We can find a sequence of step functions $\{\phi_n(x)\}$ with a compact support s.t $\int_E |f(x) - \phi_n(x)| < \frac{1}{n^2}$. So we have $\sum_{n=1}^{\infty} \int_E |f(x) - \phi_n(x)|dx = \int_E \sum_{n=1}^{\infty} |f(x) - \phi_n(x)|dx < \infty$. This implies that $\sum_{n=1}^{\infty} |f(x) - \phi_n(x)|dx < \infty$ a.e $x \in E$ hence $\phi_n(x) \xrightarrow{\text{a.e}} f(x)$ a.e $x \in E$. (This technique is the same as that of Corollary 4.19, 4.20)

□

78 (Example 4) Let $\phi(x)$ be a step function s.t $\int_{[a,b]} |f(x) - \phi(x)|dx < \frac{\epsilon}{2M}$. We do not know if $\mathrm{supp}(\phi) \subset [a, b]$. However $\phi(x) \cdot \chi_{[a,b]}$ is also a step function and its support is a subset of $[a, b]$. Therefore we may suppose that $\mathrm{supp}(\phi) \subset [a, b]$.

STEP 1.

$$\begin{aligned}\left|\int_{[a,b]} f(x)g_n(x)dx\right| &= \left|\int_{[a,b]} (f(x)-\phi(x)+\phi(x))g_n(x)dx\right| \\ &\overset{*1}{\leqq} \left|\int_{[a,b]} (f(x)-\phi(x))g_n(x)dx\right| + \left|\int_{[a,b]} \phi(x)g_n(x)dx\right| \\ &\leqq \int_{[a,b]} \left|(f(x)-\phi(x))g_n(x)\right| dx + \left|\int_{[a,b]} \phi(x)g_n(x)dx\right| \\ &\leqq M\cdot\int_{[a,b]} \left|f(x)-\phi(x)\right| dx + \left|\int_{[a,b]} \phi(x)g_n(x)dx\right| \\ &\leqq \frac{\epsilon}{2} + \left|\int_{[a,b]} \phi(x)g_n(x)dx\right|\end{aligned}$$

- (∗1) $|(f(x)-\phi(x))g_n(x)| \leqq M|f(x)-\phi(x)| \in L([a,b])$ so we may separate into two integrals.

STEP 2. Let $\phi(x) \overset{\text{def}}{=} \sum_{i=1}^{p} a_i\chi_{[x_{i-1},x_i)}(x)$ where $a = x_0 < x_1 < \cdots < x_p = b$. By assumption, it is easy to find out that $\int_{[x_{i-1},x_i)} g_n(x) \to 0$ as $n \to \infty$.

$$\begin{aligned}\int_{[a,b]} \phi(x)g_n(x)dx &= \int_{[a,b]} \sum_{i=1}^{p} a_i\chi_{[x_{i-1},x_i)}(x)\cdot g_n(x)dx \\ &= \sum_{i=1}^{p}\int_{[a,b]} a_i\chi_{[x_{i-1},x_i)}(x)\cdot g_n(x)dx \\ &= \sum_{i=1}^{p}\int_{[x_{i-1},x_i)} a_i g_n(x)dx \\ &= \sum_{i=1}^{p} a_i\int_{[x_{i-1},x_i)} g_n(x)dx\end{aligned}$$

For each $i = 1, 2\cdots, p$, when n is sufficiently large $|\int_{[x_{i-1},x_i)} g_n(x)dx| < \frac{\epsilon}{2p|a_i|}$. So we have

$$\begin{aligned}\left|\int_{[a,b]} \phi(x)g_n(x)dx\right| &\leqq \sum_{i=1}^{p} |a_i|\left|\int_{[x_{i-1},x_i)} g_n(x)dx\right| \\ &\leqq \sum_{i=1}^{p}\frac{\epsilon}{2p} = \frac{\epsilon}{2}\end{aligned}$$

Now the proof is complete.

□

79 (Example 5) Let $B \in \mathscr{M}$ be an arbitrary Lebesgue measurable set with $m(B) < \infty$.

STEP 1. Let $f_n(x) \stackrel{\text{def}}{=} \chi_A(x) \cdot \sin(\lambda_n x)$ and let $f(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} f_n(x)$. (Let us pay attention to the fact that this limit converges for all $x \in \mathbb{R}$.) Since $|f_n(x)| \leqq \chi_A(x) \in L(B)$, by Lebesgue Dominated Convergence Theorem (or Bounded Convergence Theorem), we have

$$\lim_{n\to\infty} \int_B f_n(x)dx = \int_I f(x)dx$$

STEP 2. We prove that

$$\lim_{n\to\infty} \int_B f_n(x)dx = 0.$$

We apply Corollary 4.22 to $\chi_{A\cap B}(x) \in L(\mathbb{R})$. $\forall \epsilon > 0$, we can find a step function $\phi_\epsilon(x) = \sum_{i=1}^{p} c_i \chi_{(a_{i-1},a_i]}(x)$ s.t

$$\int_{\mathbb{R}} |\chi_{A\cap B}(x) - \phi_\epsilon(x)|\, dx < \frac{\epsilon}{2}.$$

So,

$$\begin{aligned}
\left|\int_B f_n(x)dx\right| &= \left|\int_B \chi_A(x)\cdot\sin(\lambda_n x)dx\right| \\
&= \left|\int_{\mathbb{R}} \chi_{A\cap B}(x)\cdot\sin(\lambda_n x)dx\right| \\
&= \left|\int_{\mathbb{R}} (\chi_{A\cap B}(x) - \phi_\epsilon(x) + \phi_\epsilon(x))\cdot\sin(\lambda_n x)dx\right| \\
&\stackrel{*1}{\leqq} \left|\int_{\mathbb{R}} (\chi_{A\cap B}(x) - \phi_\epsilon(x))\cdot\sin(\lambda_n x)dx\right| + \left|\int_{\mathbb{R}} \phi_\epsilon(x)\cdot\sin(\lambda_n x)dx\right| \\
&\stackrel{*2}{\leqq} \int_{\mathbb{R}} |\chi_{A\cap B}(x) - \phi_\epsilon(x))\cdot\sin(\lambda_n x)|\, dx + \left|\int_{\mathbb{R}} \phi_\epsilon(x)\cdot\sin(\lambda_n x)dx\right| \\
&\stackrel{*3}{\leqq} \int_{\mathbb{R}} |\chi_{A\cap B}(x) - \phi_\epsilon(x)|\, dx + \left|\int_{\mathbb{R}} \phi_\epsilon(x)\cdot\sin(\lambda_n x)dx\right| \\
&< \frac{\epsilon}{2} + \left|\int_{\mathbb{R}} \phi_\epsilon(x)\cdot\sin(\lambda_n x)dx\right| \\
&= \frac{\epsilon}{2} + \left|\int_{\mathbb{R}} \sum_{i=1}^{p} c_i\chi_{(a_{i-1},a_i]}(x)\cdot\sin(\lambda_n x)dx\right| \\
&= \frac{\epsilon}{2} + \left|\sum_{i=1}^{p} c_i\cdot\int_{\mathbb{R}} \chi_{(a_{i-1},a_i]}(x)\cdot\sin(\lambda_n x)dx\right| \\
&= \frac{\epsilon}{2} + \left|\sum_{i=1}^{p} c_i\cdot\int_{a_{i-1}}^{a_i} \sin(\lambda_n x)dx\right| \\
&= \frac{\epsilon}{2} + \left|\sum_{i=1}^{p} c_i\cdot\frac{-(\cos\lambda_n a_i - \cos\lambda_n a_{i-1})}{\lambda_n}\right| \\
&= \frac{\epsilon}{2} + \frac{2}{\lambda_n}\sum_{i=1}^{p} |c_i| \stackrel{*4}{<} \frac{\epsilon}{2} + \frac{\epsilon}{2}
\end{aligned}$$

- $(*1)$ triangular inequality.
- $(*2)$ $\left|\int f\right| \leqq \int |f|$
- $(*3)$ $|\sin \lambda_n x| \leqq 1$
- $(*4)$ $\lambda_n \to \infty$. By taking sufficiently large n, $\cdots < \frac{\epsilon}{2}$.

So we conclude that $\lim_{n\to\infty} \left|\int_B f_n(x)dx\right| = 0$. By Step 1, we have $\int_B f(x)dx = 0$ for all $B \in \mathscr{M}$ with $m(B) < \infty$. Let $B_n = \{x \in [-n, n] \mid f(x) > 0\}$. And we have $\int_{B_n(x)} f(x)dx = 0$. So $m(B_n) = 0$. By considering $\bigcup_{n=1}^{\infty} B_n$, we have $m(\{x \in \mathbb{R} \mid f(x) > 0\}) = 0$. Similarly, $m(\{x \in \mathbb{R} \mid f(x) < 0\}) = 0$. So $f(x) = 0$ a.e $x \in \mathbb{R}$.

STEP 3. By the previous result, we have

$$\int_B (f(x))^2 dx = 0, \ \forall B \in \mathscr{M}$$

Let $B \in \mathscr{M}$ with $m(B) < \infty$. Let us pay attention to the fact that $\lim_{n\to\infty}(f_n(x))^2 = (\lim_{n\to\infty} f_n(x))^2 = (f(x))^2$.

$$\begin{aligned}
\int_B (f(x))^2 dx &= \int_B \lim_{n\to\infty} (f_n(x))^2 dx \\
&\overset{*5}{=} \lim_{n\to\infty} \int_B (f_n(x))^2 dx \\
&= \lim_{n\to\infty} \int_B \chi_A(x) \cdot \sin^2 \lambda_n x dx \\
&= \lim_{n\to\infty} \int_B \chi_A(x) \cdot \frac{1 - \cos 2\lambda_n x}{2} dx \\
&= \frac{m(A \cap B)}{2} - \lim_{n\to\infty} \frac{1}{2} \int_B \chi_A(x) \cdot \cos 2\lambda_n x dx \\
&\overset{*6}{=} \frac{m(A \cap B)}{2}
\end{aligned}$$

- $(*5)$ Lebesgue Dominated Convergence Theorem
- $(*6)$ We repeat the similar argument to prove that $\lim_{n\to\infty} \int_B \chi_A(x) \cdot \cos 2\lambda_n x dx = 0$. Let us consider $\int_{\mathbb{R}} (\chi_{A\cap B}(x) - \phi_\epsilon(x) + \phi_\epsilon(x)) \cdot \cos 2\lambda_n x dx$ ...

So $m(A \cap B) = 0$ for $\forall B \in \mathscr{M}$ with $m(B) < \infty$. Let us consider $B_n = [-n, n]$ and we have $m(\bigcup_{n=1}^{\infty} A \cap B_n) = m(A) = 0$.

$\square$

80 (Example 6) Let $F(x) \overset{\text{def}}{=} xf(x)$. $\int_{[0,1]} F(x)dx = 0$. This means that $F(x) \in L([0, 1])$ so $\int_{[0,1]} |F(x)|dx < \infty$. (Let $F(x) = 0$ if $x \notin [0, 1]$.)

STEP 1. By assumption, $\forall m \in N \cup \{0\}$, we have

$$\int_{[0,1]} x^m F(x)dx = 0.$$

Therefore $\forall P(x)$: polynomial, we have

$$\int_{[0,1]} F(x) \cdot P(x) dx = 0.$$

STEP 2. Let $\phi(x)$ be an arbitrary continuous function on $\mathbb{R}$. Then $\phi(x) \in C([0,1])$. By Weierstrass's Approximation Theorem, there exists a polynomial $P_\epsilon(x)$ s.t $\sup_{x\in[0,1]} |\phi(x) - P_\epsilon(x)| < \epsilon$ where ϵ is an arbitrary positive number.

$$\begin{aligned}
\left|\int_{\mathbb{R}} F(x)\phi(x)dx\right| &= \left|\int_{[0,1]} F(x)\phi(x)dx\right| \\
&= \left|\int_{[0,1]} F(x)(\phi(x) - P_\epsilon(x) + P_\epsilon(x))dx\right| \\
&\leqq \left|\int_{[0,1]} F(x)(\phi(x) - P_\epsilon(x))dx\right| + \left|\int_{[0,1]} F(x) \cdot P_\epsilon(x)\right| \\
&= \left|\int_{[0,1]} F(x)(\phi(x) - P_\epsilon(x))dx\right| + 0 \\
&\leqq \int_{[0,1]} |F(x)|\,|\phi(x) - P_\epsilon(x)|\,dx \\
&\leqq \int_{[0,1]} |F(x)| \cdot \epsilon dx \\
&\leqq \epsilon \cdot \int_{[0,1]} |F(x)|\,dx
\end{aligned}$$

Since $\int_{[0,1]} |F(x)|dx < \infty$, by taking $\epsilon \to 0$, we have $\int_{\mathbb{R}} F(x)\phi(x)dx = 0$ for all $\phi(x) \in C(\mathbb{R})$. By §4.3 Example 1, we have $F(x) = 0$ a.e $x \in \mathbb{R}$ hence $F(x) = 0$ a.e $x \in [0,1]$. And let us recall that $F(x) = xf(x)$, and now we conclude that $f(x) = 0$ a.e $x \in [0,1]$.

□

81 (Example 7) □

§ 4.4

82 (Darbourx Theorem)

(1) We define

$$\overline{\int_a^b} f(x)dx \overset{\text{def}}{=} \inf_{\Delta}\{\overline{S}(\Delta)\}$$

and

$$\underline{\int_a^b} f(x)dx \overset{\text{def}}{=} \sup_{\Delta}\{\underline{S}(\Delta)\},$$

where Δ is a partition of $[a,b]$.

(2) We show that $\forall\{\Delta_n\}$ a sequence of partition of $[a,b]$ with $|\Delta_n| \to 0$ we have $\overline{S}(\Delta_n) \to \overline{\int_a^b} f(x)dx$ and $\underline{S}(\Delta_n) \to \underline{\int_a^b} f(x)dx$. But the proofs are similar so we only prove $\overline{S}(\Delta_n) \to \overline{\int_a^b} f(x)dx$.

Let $\epsilon > 0$ be an arbitrary positive number. By the definition of $\overline{\int_a^b} f(x)dx$ we can find a partition Δ^* s.t

$$\overline{S}(\Delta^*) < \overline{\int_a^b} f(x)dx + \frac{\epsilon}{2}.$$

Suppose that $\Delta^* = \{x_0^*, \cdots x_K^*\}$. (In otherwords, the partition devides $[a,b]$ into K intervals.) Let $M \stackrel{\text{def}}{=} \sup_{x\in[a,b]} f(x), m \stackrel{\text{def}}{=} \inf_{[a,b]} f(x)$. ($f(x)$ is bounded on $x \in [a,b]$.) Let us consider $\Delta_n \cup \Delta^*$. (This is called refinement because the partition becomes finer by adding new partition points). We have

$$0 \leqq \overline{S}(\Delta_n) - \overline{S}(\Delta_n \cup \Delta^*) \leqq K \cdot (M-m) \cdot |\Delta_n|.$$

Seemingly this inequality seems difficult to prove but actually not. To simplify the situation, let us begin with a simpler case $\Delta_n \cup \{x^*\}$.

We add only one new partition point $\{x^*\}$. If $x_i = x^*$, then $\overline{S}(\Delta_n) - \overline{S}(\Delta_n \cup \{x^*\}) = 0$. If $x_{i-1} < x^* < x_i$, then $\overline{S}(\Delta_n) - \overline{S}(\Delta_n \cup \{x^*\}) = \sup_{a\in[x_{i-1},x_i]} f(a)(x_i - x_{i-1}) - \sup_{a\in[x_{i-1},x^*]} f(a)(x^* - x_{i-1}) - \sup_{a\in[x^*,x_i]} f(a)(x_i - x^*)$. At least $\sup_{a\in[x_{i-1},x_i]} f(a) = \sup_{a\in[x_{i-1},x^*]} f(a)$ or $\sup_{a\in[x_{i-1},x_i]} f(a) = \sup_{a\in[x^*,x_i]} f(a)$ holds. Without loss of generality, we may suppose the first case. Then $\overline{S}(\Delta_n) - \overline{S}(\Delta_n \cup \{x^*\}) = (\sup_{a\in[x_{i-1},x_i]} f(a) - \sup_{a\in[x^*,x_i]} f(a))(x_i - x^*) \leqq (M-m)|\Delta_n|$.

From the argument above, we can easily find out that if we add K points, $\overline{S}(\Delta_n) - \overline{S}(\Delta_n \cup \Delta^*) \leqq K \cdot (M-m) \cdot |\Delta_n|$. By taking sufficiently large n, $(M-m) \cdot |\Delta_n| < \frac{\epsilon}{2}$. (Here K is already fixed before $n \to \infty$.)

Finally,

$$\begin{aligned}
& \overline{S}(\Delta_n) - \overline{\int_a^b} f(x)dx \\
= \ & \left(\overline{S}(\Delta_n) - \overline{S}(\Delta_n \cup \Delta^*)\right) + \left(\overline{S}(\Delta_n \cup \Delta^*) - \overline{S}(\Delta^*)\right) + \left(\overline{S}(\Delta^*) - \overline{\int_a^b} f(x)dx\right) \\
< \ & \frac{\epsilon}{2} + \left(\overline{S}(\Delta_n \cup \Delta^*) - \overline{S}(\Delta^*)\right) + \frac{\epsilon}{2} \\
\stackrel{*}{\leqq} \ & \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

- $(*)$ $\overline{S}(\Delta_n \cup \Delta^*) - \overline{S}(\Delta^*) \leqq 0$ because $\Delta_n \cup \Delta^*$ is a refinement of Δ^*. So $\overline{S}(\Delta^*)$ is greater.

The proof for $\lim_{n\to\infty} \underline{S}(\Delta_n) \to \underline{\int_a^b} f(x)dx$ is similar.

(3) If $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$, we say that $f(x)$ is Riemann integrable on $[a,b]$.

$\square$

83 (Lemma 4.23)

STEP 1. Let $\{\Delta_n\}$ be a sequence of partition points with $|\Delta_n| \to 0$. Without loss of generality, we may suppose that $\Delta_n \subset \Delta_{n+1}$. By Darbourx theorem, $\overline{S}(\Delta_n) - \underline{S}(\Delta_n) \to \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx$.

STEP 2. Let $\Delta_n \stackrel{\text{def}}{=} \{x_0^{(n)}, \cdots, x_{k_n}^{(n)}\}, N \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} \Delta_n$ and let

$$M_i^{(n)} \stackrel{\text{def}}{=} \sup_{x \in (x_{i-1}^{(n)}, x_i^{(n)})} f(x),\ m_i^{(n)} \stackrel{\text{def}}{=} \inf_{x \in (x_{i-1}^{(n)}, x_i^{(n)})} f(x).$$

We define

$$\omega_n(x) \stackrel{\text{def}}{=} \begin{cases} M_i^{(n)} - m_i^{(n)} & x \in (x_{i-1}^{(n)}, x_i^{(n)}) \\ 0 & x \in \Delta_n \end{cases}.$$

By the assumption of $\Delta_n \subset \Delta_{n+1}$, $\omega_n(x)$ is monotone decreasing if $x \notin N$. So $\lim_{n\to\infty} \omega_n(x)$ exists. And we have

$$\begin{aligned} & \omega(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} \omega_n(x) \\ = \ & \omega_f(x) \stackrel{\text{def}}{=} \lim_{\delta\to 0} \sup_{x',x'' \in B(x,\delta)} |f(x') - f(x'')|,\ x \notin N \end{aligned}$$

Let us fix $x \in [a,b] \setminus N$. First, we prove $\omega(x) \geqq \omega_f(x)$. It is enough to prove that $\omega_n(x) \geqq \omega_f(x)$ for all $n \in \mathbb{N}$. We can find i s.t $x \in (x_{i-1}^{(n)}, x_i^{(n)})$. We can always take $\delta > 0$ s.t $B(x,\delta) \subset (x_{i-1}^{(n)}, x_i^{(n)})$. Therefore $\omega_n(x) = M_i^{(n)} - m_i^{(n)} \geqq \sup_{x',x'' \in B(x,\delta)} |f(x') - f(x'')| \geqq \omega_f(x)$.

Next, we prove $\omega(x) \leqq \omega_f(x)$. It is enough to prove that $\omega(x) \leqq \sup_{x',x'' \in B(x,\delta)} |f(x') - f(x'')|$ for all $\delta > 0$. For all $\delta > 0$, we can find $n \in N$ and i s.t $x \in (x_{i-1}^{n}, x_i^{(n)}) \subset B(x,\delta)$, we have $\omega(x) \leqq \omega_n(x) = M_i^{(n)} - m_i^{(n)} \leqq \sup_{x',x'' \in B(x,\delta)} |f(x') - f(x'')|$

Since $m(N) = 0$, we conclude that $\omega_n(x) \to \omega_f(x)$ a.e $x \in [a,b]$.

STEP 3. Since $\omega_n(x) \leqq \sup_{x \in [a,b]} f(x) - \inf_{x \in [a,b]} f(x) = M - m < \infty$ ($f(x)$ is bounded.), by Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_{[a,b]} \omega_n(x)dx = \int_{[a,b]} \omega_f(x)dx.$$

It is easy to verify that the left hand side is $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)$.

$$
\begin{aligned}
\int_{[a,b]} \omega_n(x)dx &= \int_{[a,b]} \sum_{i=1}^{k_n} (M_i^{(n)} - m_i^{(n)}) \chi_{(x_{i-1}^{(n)}, x_i^{(n)})}(x)dx \\
&= \sum_{i=1}^{k_n} \int_{[a,b]} (M_i^{(n)} - m_i^{(n)}) \chi_{(x_{i-1}^{(n)}, x_i^{(n)})}(x)dx \\
&= \sum_{i=1}^{k_n} (M_i^{(n)} - m_i^{(n)}) m((x_{i-1}^{(n)}, x_i^{(n)})) \\
&= \sum_{i=1}^{k_n} (M_i^{(n)} - m_i^{(n)})(x_i^{(n)} - x_{i-1}^{(n)}) \\
&= \sum_{i=1}^{k_n} M_i^{(n)}(x_i^{(n)} - x_{i-1}^{(n)}) - \sum_{i=1}^{k_n} m_i^{(n)}(x_i^{(n)} - x_{i-1}^{(n)}) \\
&= \overline{S}(\Delta_n) - \underline{S}(\Delta_n) \to \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)
\end{aligned}
$$

Now the proof is complete.

□

84 (Theorem 4.24) $f(x)$ is Riemann integrable on $[a, b] \Leftrightarrow \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx = 0 \Leftrightarrow \int_{[a,b]} \omega_f(x)dx = 0 \Leftrightarrow \omega_f(x) = 0$ a.e $x \in [a, b] \Leftrightarrow f(x) = 0$ is continuous at a.e $x \in [a, b]$. Therefore $f(x)$ is Riemann integrable if and only $m(D) = 0$ where D is a set of points of discontinuity of $f(x)$. □

85 (Theorem 4.25)

STEP 1. ($f(x)$ is Lebesgue measurable) By the conclusion of Theorem 4.24, $f(x)$ is continuous almost everywhere $x \in [a, b]$. Let D be the set of discontinuity of $f(x)$. D is a measure zero set. ($D \in \mathscr{M}$) Then

$$
\begin{aligned}
& \{x \in [a, b] \mid f(x) > t\} \\
= \; & \{x \in [a, b] \mid f(x) > t\} \setminus D \cup \{x \in [a, b] \mid f(x) > t\} \cap D \\
= \; & [a, b] \setminus D \cap G \cup \{x \in [a, b] \mid f(x) > t\} \cap D
\end{aligned}
$$

where G is an open set. ($f(x)$ is continuous on $[a, b] \setminus D$. $G \in \mathscr{M}$) Since D is a measure zero set, $\{x \in [a, b] \mid f(x) > t\} \cap D \subset D$ so $\{x \in [a, b] \mid f(x) > t\} \cap D$ is also a measure zero set hence measurable.

STEP 2. ($f(x)$ is Lebesgue integrable) Since $|f(x)| \leqq M < \infty$, ($\because f(x)$ is bounded on $[a, b]$ by assumption), $f(x) \in L([a, b])$

STEP 3. ((L) $\int_{[a,b]} f(x)dx$ = (R) $\int_a^b f(x)dx$) Let us pick a sequence of partitions of the interval $[a,b]$ $\{\Delta_n\}_{n\geqq 1}$ with $|\Delta_n| \to 0$. (We use the same notations as the previous lemma and the theorems.) Since

$$\sum_{i=1}^{k_n} m_i^{(n)} \cdot \chi_{(x_{i-1}^{(n)}, x_i^{(n)})}(x) \leqq f(x) \leqq \sum_{i=1}^{k_n} M_i^{(n)} \cdot \chi_{(x_{i-1}^{(n)}, x_i^{(n)})}(x),$$

by taking integral of them, we have

$$\sum_{i=1}^{k_n} m_i^{(n)} \cdot m((x_{i-1}^{(n)}, x_i^{(n)})) \leqq \text{(L)} \int_{[a,b]} f(x)dx \leqq \sum_{i=1}^{k_n} M_i^{(n)} \cdot m((x_{i-1}^{(n)}, x_i^{(n)})).$$

The left hand side and the right hand side are $\underline{S}(\Delta_n)$ and $\overline{S}(\Delta_n)$ respectively, so we have

$$\underline{S}(\Delta_n) \leqq \text{(L)} \int_{[a,b]} f(x)dx \leqq \overline{S}(\Delta_n)$$

Since $f(x)$ is Riemann integrable, so $\underline{S}(\Delta_n), \overline{S}(\Delta_n) \to$ (R) $\int_a^b f(x)dx$ as $n \to \infty$. So we have the desired result.

□

86 (Exercise 1) See Exercise 27 in §Exercise. $\chi_E(x), E \subset [0,1]$ is Riemann integrable if and only if $m(\overline{E} \setminus \overset{\circ}{E}) = 0$. If F is a closed set, $\overline{F} = F$. So $m(\overline{F} \setminus \overset{\circ}{F}) = m(F \setminus \overset{\circ}{F}) \leqq m(F) = 0$. Hence $\chi_F(x)$ is Riemann integrable. □

87 (Exercise 2) Let D_1, D_2 be sets of discontinuity of $f(x)$ and $g \circ f(x)$ respectively. If $f(x)$ is continuous at x_0 then $g \circ f(x)$ is also continuous at x_0. So if $g \circ f(x)$ is not continuous at x_0, then $f(x)$ is not continuous at x_0. Therefore $D_2 \subset D_1$. D_1 is a measure zero set so is D_2. This implies that $g \circ f(x)$ is also Riemann integrable. □

88 (Exercise 3) Let D_1, D_2 be sets of discontinuity of $f(x), g(x)$ respectively. Then D_1, D_2 are measure zero set. Now let us pick an arbitrary point $x_0 \in [a,b] \setminus (D_1 \cup D_2)$. Since $x_0 \in [a,b] = \overline{E}$, therefore there exists $\{x_n\}_{n\geqq 1} \subset E$ s.t $x_n \to x_0$. (You may consider $x_0 \in E$ or $x_0 \in E'$. In any case, we can find $\{x_n\} \subset E$ s.t $x_n \to x_0$. Here we allow $\{x_n\}$ to contain the same points.) Moreover since $x_0 \notin D_1$ and $x_0 \notin D_2$, we have $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$. By assumption $f(x_n) = g(x_n)$. So $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) \Rightarrow f(x_0) = g(x_0)$. We conclude that $f(x) = g(x)$ for all $x \in [a,b] \setminus (D_1 \cup D_2)$ hence $f(x) = g(x)$ a.e $x \in [a,b]$. Now we have the desired conclusion. □

89 (Theorem 4.26) As we have stated the proof is quite easy. Since $|f(x)| \cdot \chi_{E_k}(x) \leqq |f(x)| \cdot \chi_{E_{k+1}}(x)$, by monotone convergence theorem, we have

$$\lim_{k\to\infty} \int_E |f(x)| \cdot \chi_{E_k}(x)dx = \int_E \lim_{k\to\infty} |f(x)| \cdot \chi_{E_k}(x)dx.$$

The left hand side is

$$\lim_{k\to\infty} \int_E |f(x)| \cdot \chi_{E_k}(x)dx = \lim_{k\to\infty} \int_{E_k} |f(x)|\, dx < \infty.$$

by the properties of Lebesgue integral of non-negative measurable functions. The right hand side is

$$\int_E \lim_{k\to\infty} |f(x)| \cdot \chi_{E_k}(x)dx = \int_E |f(x)|\, dx$$

because if $E_k \to E$ then $\chi_{E_k}(x) \to \chi_E(x)$. (We have proven this before.) So we conclude that $\int_E |f(x)| < \infty$. Finally since

$$|f(x) \cdot \chi_{E_k}(x)| \leqq |f(x)| \in L(E),$$

by Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned} \lim_{k\to\infty} \int_E f(x) \cdot \chi_{E_k}(x)dx &= \int_E \lim_{k\to\infty} f(x) \cdot \chi_{E_k}(x)dx \\ &= \int_E f(x) \cdot \chi_E(x)dx = \int_E f(x)dx. \end{aligned}$$

The left hand side is

$$\lim_{k\to\infty} \int_E f(x) \cdot \chi_{E_k}(x)dx = \lim_{k\to\infty} \int_{E_k} f(x)dx$$

So the proof is complete. □

90 (Example 1) Let $f(x) = \frac{\sin x}{x}$. We prove that the Riemann improper integral of $f(x)$ is finite, however the Lebesgue integral of $|f(x)|$ is inifinity.

STEP 1.

$$(\mathrm{R}) \int_{(0,\infty)} f(x)dx = \lim_{t\to\infty} (\mathrm{R}) \int_0^t \frac{\sin x}{x}dx.$$

Let $a(t) \overset{\text{def}}{=} (\mathrm{R}) \int_0^t \frac{\sin x}{x}dx$. We prove that $\lim_{t_1<t_2\to\infty} |a(t_1) - a(t_2)| = 0$. We can find $k \leqq \ell \in \mathbb{N}$ s.t $2(k-1)\pi < t_1 \leqq 2k\pi \leqq 2\ell\pi \leqq t_2 < 2(\ell+1)\pi$.

$$\begin{aligned} |a(t_1) - a(t_2)| &= \left| \int_{t_1}^{t_2} \frac{\sin x}{x}dx \right| \\ &= \left| \int_{t_1}^{2k\pi} \frac{\sin x}{x}dx + \int_{2k\pi}^{2\ell\pi} \frac{\sin x}{x}dx + \int_{2\ell\pi}^{t_2} \frac{\sin x}{x}dx \right| \\ &\leqq \left| \int_{t_1}^{2k\pi} \frac{\sin x}{x}dx + \int_{2k\pi}^{2\ell\pi} \frac{\sin x}{x}dx + \int_{2\ell\pi}^{t_2} \frac{\sin x}{x}dx \right| \\ &\leqq \int_{t_1}^{2k\pi} \frac{|\sin x|}{x}dx + \left| \int_{2k\pi}^{2\ell\pi} \frac{\sin x}{x}dx \right| + \int_{2\ell\pi}^{t_2} \frac{|\sin x|}{x}dx \\ &\leqq \int_{t_1}^{2k\pi} \frac{|\sin x|}{t_1}dx + \left| \int_{2k\pi}^{2\ell\pi} \frac{\sin x}{x}dx \right| + \int_{2\ell\pi}^{t_2} \frac{|\sin x|}{t_1}dx \\ &\leqq \int_{t_1}^{2k\pi} \frac{1}{t_1}dx + \left| \int_{2k\pi}^{2\ell\pi} \frac{\sin x}{x}dx \right| + \int_{2\ell\pi}^{t_2} \frac{1}{t_1}dx \\ &= \frac{(2k\pi - t_1)}{t_1} + \left| \int_{2k\pi}^{2\ell\pi} \frac{\sin x}{x}dx \right| + \frac{(t_2 - 2\ell\pi)}{t_1} \\ &\leqq \frac{2\pi}{t_1} + \left| \int_{2k\pi}^{2\ell\pi} \frac{\sin x}{x}dx \right| + \frac{2\pi}{t_1} \end{aligned}$$

Since $\frac{2\pi}{t_1} \to 0$ as $t_1 \to \infty$, it is enough for us to prove that

$$\lim_{k,\ell\to\infty} \int_{2k\pi}^{2\ell\pi} \frac{\sin x}{x} dx = 0.$$

It is not difficult to verify that $\int_{2k\pi}^{2\ell\pi} \frac{\sin x}{x} dx \geqq 0$ because

$$\begin{aligned}
& \sum_{m=0}^{\ell-k-1} \int_{(2k+2m)\pi}^{(2k+2m+2)\pi} \frac{\sin x}{x} dx \\
= & \sum_{m=0}^{\ell-k-1} \left(\int_{(2k+2m)\pi}^{(2k+2m+1)\pi} \frac{\sin x}{x} dx + \int_{(2k+2m+1)\pi}^{(2k+2m+2)\pi} \frac{\sin x}{x} dx \right) \\
\geqq & \sum_{m=0}^{\ell-k-1} \left(\int_{(2k+2m)\pi}^{(2k+2m+1)\pi} \frac{\sin x}{(2k+2m+1)\pi} dx + \int_{(2k+2m+1)\pi}^{(2k+2m+2)\pi} \frac{\sin x}{(2k+2m+1)\pi} dx \right) = 0.
\end{aligned}$$

We separate each term into two parts. ($\sin x \geqq 0$ and $\sin x \leqq 0$). Next,

$$\begin{aligned}
0 \leqq & \sum_{m=0}^{\ell-k-1} \int_{(2k+2m)\pi}^{(2k+2m+2)\pi} \frac{\sin x}{x} dx \\
= & \sum_{m=0}^{\ell-k-1} \left(\int_{(2k+2m)\pi}^{(2k+2m+1)\pi} \frac{\sin x}{x} dx + \int_{(2k+2m+1)\pi}^{(2k+2m+2)\pi} \frac{\sin x}{x} dx \right) \\
\leqq & \sum_{m=0}^{\ell-k-1} \left(\int_{(2k+2m)\pi}^{(2k+2m+1)\pi} \frac{\sin x}{(2k+2m)\pi} dx + \int_{(2k+2m+1)\pi}^{(2k+2m+2)\pi} \frac{\sin x}{(2k+2m+2)\pi} dx \right) \\
= & \sum_{m=0}^{\ell-k-1} \left(\frac{2}{(2k+2m)\pi} - \frac{2}{(2k+2m+2)\pi} \right) \\
= & \sum_{m=0}^{\ell-k-1} \left(\frac{4\pi}{(2k+2m)(2k+2m+2)\pi^2} \right) \\
= & \sum_{m=0}^{\ell-k-1} \left(\frac{1}{(k+m)(k+m+1)\pi} \right) \\
= & \sum_{m=k}^{\ell-1} \left(\frac{1}{m\cdot(m+1)\pi} \right) \\
< & \sum_{m=k}^{\infty} \left(\frac{1}{m\cdot(m+1)\pi} \right) = \frac{1}{k\pi} \to 0 \text{ as } k \to \infty
\end{aligned}$$

Therefore $|a(t_1) - a(t_2)|$ is a Cauchy sequence. Hence $a(t)$ converges.

STEP 2. Next we prove that $\frac{|\sin x|}{x} \notin L([0,\infty))$

$$
\begin{aligned}
(\mathrm{L}) \int_0^\infty \frac{|\sin x|}{x} dx &= \sum_{k=1}^{\infty} (\mathrm{L}) \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx \\
&\geqq \sum_{k=0}^{\infty} (\mathrm{L}) \int_{(k+\frac{1}{6})\pi}^{(k+\frac{5}{6})\pi} \frac{|\sin x|}{x} dx \\
&\geqq \sum_{k=0}^{\infty} (\mathrm{L}) \int_{(k+\frac{1}{6})\pi}^{(k+\frac{5}{6})\pi} \frac{1}{2} \cdot \frac{1}{x} dx \\
&\geqq \sum_{k=0}^{\infty} (\mathrm{L}) \int_{(k+\frac{1}{6})\pi}^{(k+\frac{5}{6})\pi} \frac{1}{2} \cdot \frac{1}{(k+\frac{5}{6})\pi} dx \\
&= \sum_{k=0}^{\infty} \frac{1}{2} \cdot \frac{1}{(k+\frac{5}{6})\pi} \cdot \frac{4\pi}{6} \\
&= \frac{1}{3} \cdot \sum_{k=0}^{\infty} \frac{1}{(k+\frac{5}{6})} \\
&= \frac{1}{3} \cdot \sum_{k=1}^{\infty} \frac{1}{k} = \infty
\end{aligned}
$$

□

91 (Example 3)

STEP 1. Let us consider

$$\int_{(0,1)} \frac{-\ln x}{1-x} dx.$$

Since $\forall x \in (0,1)$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. So

$$
\begin{aligned}
\int_{(0,1)} \frac{-\ln x}{1-x} dx &= \int_{(0,1)} (-\ln x) \sum_{n=0}^{\infty} x^n dx \\
&\overset{*1}{=} \sum_{n=0}^{\infty} \int_{(0,1)} (-\ln x) \cdot x^n dx \\
&= \sum_{n=0}^{\infty} \int_{(0,1]} (-\ln x) \cdot x^n dx
\end{aligned}
$$

- (∗1) By a corollary of monotone convergence theorem. (Theorem 4.6)

STEP 2. We find

$$\int_{(0,1]} (-\ln x) \cdot x^n dx.$$

By monotone convergence theorem, we have

$$\lim_{\epsilon \to +0} \int_{[\epsilon,1]} (-\ln x) \cdot x^n dx.$$

Since $\int_{[\epsilon,1]}(-\ln x)\cdot x^n dx$ is Riemann integrable (because the function is continuous on $[\epsilon,1]$), (R) $\int_{[\epsilon,1]}(-\ln x)\cdot x^n dx =$ (L) $\int_{[\epsilon,1]}(-\ln x)\cdot x^n dx$. So let us we find

$$\lim_{\epsilon\to+0} \text{(R)} \int_{[\epsilon,1]} (-\ln x)\cdot x^n dx.$$

By integration by substitution (let $x = e^{-t}, u = (n+1)t$), we have

$$\begin{aligned}
& \lim_{\epsilon\to+0} \text{(R)} \int_{[\epsilon,1]} (-\ln x)\cdot x^n dx \\
= & \lim_{\epsilon\to+0} \text{(R)} \int_{[0,-\ln\epsilon]} t\cdot e^{-(n+1)t} dt \\
= & \lim_{\epsilon\to+0} \text{(R)} \int_{[0,-(n+1)\ln\epsilon]} \frac{u}{(n+1)^2}\cdot e^{-u} dt \\
= & \text{(R)} \int_{[0,\infty)} \frac{u}{(n+1)^2}\cdot e^{-u} dt \\
\overset{*2}{=} & \frac{1}{(n+1)^2}
\end{aligned}$$

- ($*2$) Let us recall the definition of Gamma function. $\Gamma(\alpha) \overset{\text{def}}{=} \int_{[0,\infty)} x^{\alpha-1}e^{-x}dx$. $\Gamma(n) = (n-1)!$ if $n \in \mathbb{N}$.

STEP 3. Finally,

$$\int_{(0,1)} \frac{-\ln x}{1-x} dx = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \overset{*3}{=} \frac{\pi^2}{6}.$$

- ($*3$) This is a well-known fact. We use this fact without proof.

So $I = \frac{-\pi^2}{6}$.

□

92 (Notice) □

93 (Exercise 4) We prove that $\int_{[0,\infty)} |\sin x^2| dx = \infty$.

$$
\begin{aligned}
\text{(L)} \int_{[0,\infty)} \left|\sin x^2\right| dx &= \sum_{n=0}^{\infty} \text{(L)} \int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \left|\sin x^2\right| dx \\
&= \sum_{n=0}^{\infty} \text{(R)} \int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \left|\sin x^2\right| dx \\
&\overset{*1}{=} \sum_{n=0}^{\infty} \text{(R)} \frac{1}{2} \int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{\sqrt{t}} dt \\
&= \sum_{n=0}^{\infty} \text{(L)} \frac{1}{2} \int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{\sqrt{t}} dt \\
&\geqq \sum_{n=0}^{\infty} \text{(L)} \frac{1}{2} \int_{(n+\frac{1}{6})\pi}^{(n+\frac{5}{6})\pi} \frac{|\sin t|}{\sqrt{t}} dt \\
&\geqq \sum_{n=0}^{\infty} \text{(L)} \frac{1}{2} \int_{(n+\frac{1}{6})\pi}^{(n+\frac{5}{6})\pi} \frac{1}{2\sqrt{t}} dt \\
&\geqq \sum_{n=0}^{\infty} \text{(L)} \frac{1}{2} \int_{(n+\frac{1}{6})\pi}^{(n+\frac{5}{6})\pi} \frac{1}{2\sqrt{(n+\frac{5}{6}\pi)}} dt \\
&= \sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{4\pi}{6} \cdot \frac{1}{2\sqrt{(n+\frac{5}{6}\pi)}} \\
&= \sum_{n=0}^{\infty} \frac{\pi}{6} \cdot \frac{1}{\sqrt{(n+\frac{5}{6}\pi)}} = \infty
\end{aligned}
$$

- (∗1) Let us regard the integral as a Riemann integral and do integration by substitution. We still do not know whether we can do integration by substitution in Lebesgue integral.

□

§ 4.5

94 (Lemma 4.28)

(1) $(af(x,y) \in \mathscr{F})$

STEP 1. (a) If $y \mapsto f(x,y)$ is non-negative measurable, then $y \mapsto a \cdot f(x,y)$ is also non-negative measurable on $\mathbb{R}^q$.

STEP 2. (b) If $F(x) \overset{\text{def}}{=} \int_{\mathbb{R}^q} f(x,y)dy$ is non-negative measurable, so is $aF(x)$. Since $a \cdot F(x) = \int_{\mathbb{R}^q} a \cdot f(x,y)dy$, $af(x,y)$ also satiefies (b).

STEP 3. (c) Since $\int_{\mathbb{R}^p} F(x)dx = \int_{\mathbb{R}^d} f(x,y)dxdy$, we have

$$a \cdot \int_{\mathbb{R}^p} F(x)dx = a \cdot \int_{\mathbb{R}^d} f(x,y)dxdy.$$

By linearity of integral we have

$$\int_{\mathbb{R}^p} a \cdot F(x)dx = \int_{\mathbb{R}^d} a \cdot f(x,y)dxdy.$$

Since $a \cdot F(x) = a \cdot \int_{\mathbb{R}^q} f(x,y)dy = \int_{\mathbb{R}^q} a \cdot f(x,y)dy$, by substituting this to the formula above, we have

$$\int_{\mathbb{R}^p} \left(\int_{\mathbb{R}^q} a \cdot f(x,y)dy \right) dx = \int_{\mathbb{R}^d} a \cdot f(x,y)dxdy.$$

(2) $(f_1(x,y) + f_2(x,y) \in \mathscr{F})$

STEP 1. (a) Suppose N_1, N_2 are measure zero sets and if $x_i \notin N_i$ then $y \mapsto f_i(x,y)$ is non-negative measurable on $\mathbb{R}^q$. $(i = 1,2)$. $x \notin N_1 \cup N_2$ $(m(N_1 \cup N_2) = 0)$ then both $y \mapsto f_1(x,y), y \mapsto f_2(x,y)$ are non-negative measurable on $\mathbb{R}^q$ so we have $y \mapsto f_1(x,y) + f_2(x,y)$ is non-negative measurable on $\mathbb{R}^q$. So $f_1(x,y) + f_2(x,y)$ satisfies (a).

STEP 2. (b) Let $F_1(x) \overset{\text{def}}{=} \int_{R^q} f_1(x,y)dy$ and let $F_2(x) \overset{\text{def}}{=} \int_{R^q} f_2(x,y)dy$. By assumption, $F_1(x), F_2(x)$ are non-negative measurable functions on $\mathbb{R}^p$. So $F_1(x)+F_2(x)$ is also non-negative measurable functions. Moreover by linearity of integral of non-negative measurable functions, we have $F_1(x) + F_2(x) = \int_{\mathbb{R}^q} (f_1(x) + f_2(x))\, dx$. So $f_1(x) + f_2(x)$ satisfies (b).

STEP 3. (c) By assumption, $\int_{\mathbb{R}^p} F_1(x)dx = \int_{\mathbb{R}^d} f_1(x,y)dxdy$ and $\int_{\mathbb{R}^p} F_2(x)dx = \int_{\mathbb{R}^d} f_2(x,y)dxdy$. Therefore, $\int_{\mathbb{R}^p} F_1(x)dx+\int_{\mathbb{R}^p} F_2(x)dx = \int_{\mathbb{R}^d} f_1(x,y)dxdy+\int_{\mathbb{R}^d} f_2(x,y)dxdy$. Since integrals of non-negative measurable functions have linearity so we have

$$\int_{\mathbb{R}^p} (F_1(x) + F_2(x))\, dx = \int_{\mathbb{R}^d} (f_1(x,y) + f_2(x,y))\, dxdy.$$

So $f_1(x,y) + f_2(x,y)$ satisfies (c).

(3) $(f(x,y) - g(x,y) \in \mathscr{F})$

STEP 1. This is just a review. If $f(x), g(x)$ are measurable function on $E \in \mathscr{M}$ and $f(x) - g(x)$ is well-defined (i.e $\infty - \infty$ does not happen.), then $f(x) - g(x)$ is also measurable. (See Chapter 3.)

Now suppose that if $f(x), g(x)$ are measurable on $E \in \mathscr{M}$ and $f(x) - g(x)$ is defined a.e $x \in E$ then $f - g$ is measurable on E. (i.e $\infty - \infty$ happens but it happens only at x in a measure zero set.) There exists $N \subset E$ and $m(N) = 0$ and $f(x) - g(x)$ is well-defined on $E \setminus N$. Let us consider $\{x \in E \mid f(x) - g(x) > t\} = \{x \in E \mid f(x) - g(x) > t\} \setminus N \cup \{x \in E \mid f(x) - g(x) > t\} \cap N$. $\{x \in E \mid f(x) - g(x) > t\} \setminus N = \{x \in E \setminus N \mid f(x) - g(x) > t\} \in \mathscr{M}$ because we may regard $f(x), g(x)$ as measurable functions defined on $E \setminus N$, and $f(x) - g(x)$ is defined on $E \setminus N$ so $f(x) - g(x)$ is measurable on $E \setminus N$. And $\{x \in E \mid f(x) - g(x) > t\} \cap N \subset N \in \mathscr{M}$. So the proof is complete.

STEP 2. Let $F_1(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^q} f(x,y)dy$ and $F_2(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^q} g(x,y)dy$. Since $\int_{\mathbb{R}^p} F_2(x)dx < \infty$, $F_2(x) < \infty$ a.e $x \in \mathbb{R}^p$. And if $F_2(x) = \int_{\mathbb{R}^q} g(x,y)dy < \infty$ (here x is fixed.), $y \mapsto g(x,y) < \infty$ a.e $y \in \mathbb{R}^q$.

Let $P(x) \stackrel{\text{def}}{=}$ "$y \mapsto g(x,y) < \infty$ a.e $y \in \mathbb{R}^q$". From the argument above, we conclude that the proposition $P(x)$ is true a.e $x \in \mathbb{R}^p$.

(In the argument above, let us recall the fact that when f is non-negative $\int_E f < \infty \Rightarrow f < \infty$ a.e $x \in E$.)

STEP 3. (a) By assumption for a.e $x \in \mathbb{R}^p$, $y \mapsto f(x,y)$ and $y \mapsto g(x,y)$ are non-negative and measurable. We know that $y \mapsto f(x,y) - g(x,y)$ is non-negative by assumption. We still need to prove that $y \mapsto f(x,y) - g(x,y)$ is measurable a.e $x \in \mathbb{R}^p$. However it is enough for us to prove that $y \mapsto f(x,y) - g(x,y)$ is well-defined a.e $x \in \mathbb{R}^p$.

Let us fix $x \in \mathbb{R}^p$ where $P(x)$ is true. Since $y \mapsto g(x,y) < \infty$ a.e $y \in \mathbb{R}^q$, $y \mapsto f(x,y) - g(x,y)$ is well-defined a.e $y \in \mathbb{R}^q$. (i.e $\infty - \infty$ does not happen.) Therefore $y \mapsto f(x,y) - g(x,y)$ is measurable. So $y \mapsto f(x,y) - g(x,y)$ is measurable a.e $x \in \mathbb{R}^p$.

STEP 4. (b) $F_1(x) - F_2(x)$ is well-defined a.e $x \in \mathbb{R}^p$ because $F_2(x) < \infty$ a.e $x \in \mathbb{R}^p$. $F_1(x), F_2(x)$ are measurable on $\mathbb{R}^p$, so $F_1(x) - F_2(x)$ is also measurable.

STEP 5. (c) Since $g(x,y) \in L(\mathbb{R}^d)$, $\int_{\mathbb{R}^p} F_2(x)dx = \int_{\mathbb{R}^d} g(x,y)dxdy < \infty$. (finite) Therefore, we may subtract it from $\int_{\mathbb{R}^p} F_1(x)dx = \int_{\mathbb{R}^d} f(x,y)dxdy$. (We just want to avoid $\infty - \infty$.) So $\int_{\mathbb{R}^p} F_1(x)dx - \int_{\mathbb{R}^p} F_2(x)dx = \int_{\mathbb{R}^d} f(x,y)dxdy - \int_{\mathbb{R}^d} g(x,y)dxdy$. By Theorem 4.10, this implies that $\int_{\mathbb{R}^p} (F_1(x) - F_2(x))\, dx = \int_{\mathbb{R}^d} (f(x,y) - g(x,y))\, dxdy$.

(4) $(f(x,y) \in \mathscr{F})$

STEP 1. (a) By assumption, there exists $\{N_k\}_{k \geqq 1}$ with $m_p(N_k) = 0$ for all $k \in \mathbb{N}$ s.t $\forall k \in \mathbb{N}, \forall x \notin N_k, y \mapsto f_k(x,y)$ is a measurable function on $\mathbb{R}^q$. Let $N \stackrel{\text{def}}{=} \bigcup_{k=1}^{\infty} N_k$. Then $m_p(N) = 0$. $\forall x \notin N, y \mapsto f_k(x,y)$ is a measurable function on $\mathbb{R}^q$ for all $k \in \mathbb{N}$. So if $x \notin \mathbb{N}$, then $y \mapsto \lim_{k\to\infty} f_k(x,y)$ is a measurable function. This means that for a.e $x \in \mathbb{R}^p$, $y \mapsto \lim_{k\to\infty} f_k(x,y)(= f(x,y))$ is a measurable function. Obviously $f(x,y) \geqq 0$. Now the proof is complete.

STEP 2. (b) Let $F_k(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^q} f_k(x,y)dy$ and let $F(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^q} f(x,y)dy$. By assumption, $F_k(x)$ is a non-negative measurable function on $\mathbb{R}^p$ for all $k \in \mathbb{N}$. Since $f(x,y)$ is a measurable funtion for a.e $x \in \mathbb{R}^p$, $F(x)$ is defined a.e $x \in \mathbb{R}^p$. By monotone convergence theorem, if $x \notin N$,

$$\begin{aligned} \lim_{k\to\infty} F_k(x) &= \int_{\mathbb{R}^q} \lim_{k\to\infty} f_k(x,y)dy \\ &= \int_{\mathbb{R}^q} f(x,y)dy = F(x). \end{aligned}$$

So $\lim_{k\to\infty} F_k(x) = F(x)$ a.e $x \in \mathbb{R}^p$. Since $\lim_{k\to\infty} F_k(x)$ is measurable on $\mathbb{R}^p$ (because the limit of a sequence of measurable functions is also measurable), $F(x)$ is also measurable on $\mathbb{R}^p$. Obviously $F(x)$ is non-negative.

STEP 3. (c) By assumption,

$$\int_{\mathbb{R}^q} F_k(x)dx = \int_{\mathbb{R}^d} f_k(x,y)dxdy.$$

By taking $k \to \infty$, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^q} F_k(x)dx = \lim_{k\to\infty} \int_{\mathbb{R}^d} f_k(x,y)dxdy.$$

(The sequence is of integrals is monotone increasing, so the limits exist.) By monotone convergence theorem,

$$\int_{\mathbb{R}^q} \lim_{k\to\infty} F_k(x)dx = \int_{\mathbb{R}^d} \lim_{k\to\infty} f_k(x,y)dxdy.$$

Since $\lim_{k\to\infty} F_k(x) = F(x)$ a.e $x \in \mathbb{R}^p$ and $f_k(x,y) \to f(x,y)$, we have

$$\int_{\mathbb{R}^q} F(x)dx = \int_{\mathbb{R}^d} f(x,y)dxdy.$$

Now the proof is complete.

(5) ($f(x,y) \in \mathscr{F}$) Let us consider $g_k(x,y) = f_1(x,y) - f_k(x,y) \geqq 0$. Since $f_k(x) \in L(\mathbb{R}^d)$, we have $g_k(x,y) \in \mathscr{F}$ by (3). Moreover $g_k(x,y) \nearrow f_1(x,y) - f(x,y)$. So $f_1(x,y) - f(x,y) \in \mathscr{F}$ by (4). Finally, $f(x,y) = f_1(x,y) - (f_1(x,y) - f(x,y)) = f(x,y) \in \mathscr{F}$ by (3). ($f_1(x,y) - f(x,y) \leqq f_1(x,y) \in L(\mathbb{R}^d)$)

□

95 (Theorem 4.27) First we prove that $f(x,y) \overset{\text{def}}{=} \chi_E(x,y) \in \mathscr{F}$ for all $E \in \mathscr{M}_d$. However, we can not prove this directly. So we first start with $E = I_1 \times I_2$ where I_1, I_2 are half open rectangles on $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively.

(1) ($E = I_1 \times I_2$ where I_1, I_2 are half open rectangles.)

STEP 1. (*a*)

$$y \mapsto f(x,y) \overset{\text{def}}{=} \begin{cases} \chi_{I_2}(y) & x \in I_1 \\ 0 & x \notin I_1 \end{cases}$$

From this, we can find out that $y \mapsto f(x,y)$ is a measurable fuction on $\mathbb{R}^q$ for all $x \in \mathbb{R}^p$.

STEP 2. (*b*)

$$F(x) \overset{\text{def}}{=} \int_{\mathbb{R}^q} \chi_{I_1}(x)\chi_{I_2}(y)dy = m_q(I_2) \cdot \chi_{I_1}(x).$$

is a measurable function on $\mathbb{R}^p$.

STEP 3. (*c*)

$$\int_{\mathbb{R}^p} F(x)dx = \int_{\mathbb{R}^p} m_q(I_2) \cdot \chi_{I_1}(x)dx = m_p(I_1) \cdot m_q(I_2)$$

and

$$\int_{\mathbb{R}^d} f(x,y)dxdy = \int_{\mathbb{R}^d} \chi_{I_1\times I_2}(x,y)dxdy = m_d(I_1 \times I_2).$$

We claim that $m_p(I_1) \cdot m_q(I_2) = m_d(I_1 \times I_2)$. Suppose that $I_1 = \prod_{i=1}^p (a_{1,i}, b_{1,i}]$ and $I_2 = \prod_{j=1}^q (a_{2,j}, b_{2,j}]$. Then $I_1 \times I_2 = \prod_{i=1}^p (a_{1,i}, b_{1,i}] \times \prod_{j=1}^q (a_{2,j}, b_{2,j}]$ is also a half open rectangle. Since $m(I) = |I|$, both $m_p(I_1) \cdot m_q(I_2)$ and $m_d(I_1 \times I_2)$ are $\prod_{i=1}^p (b_{1,i} - a_{1,i}) \cdot \prod_{j=1}^q (b_{2,j} - a_{2,j})$.

(2) ($E \in \mathscr{O}^d$) In Chapter 1, we proved that an open set is decomposed into a disjoint union of half open rectangles. We may suppose that $E = \bigcup_{k=1}^{\infty} I_k$ where $\{I_k\}_{k \geqq 1}$ are disjoint half open rectangles on $\mathbb{R}^d$. Let $E_n \stackrel{\text{def}}{=} \bigcup_{k=1}^{n} I_k$. Let $f_n(x,y) \stackrel{\text{def}}{=} \chi_{E_n}(x,y) = \sum_{k=1}^{n} \chi_{I_k}(x,y)$. Each $\chi_{I_k}(x,y) \in \mathscr{F}$ and by Lemma 4.27, $f_n(x,y) = \sum_{k=1}^{n} \chi_{I_k}(x,y) \in \mathscr{F}$. Since $f_n(x) \nearrow \chi_E(x,y)$. Again by Lemma 4.27, we have $\chi_E(x,y) \in \mathscr{F}$.

(3) (E is a bounded closed set) We can find $0 < r < \infty$ s.t $E \subset B(0,r)$. Let $G_1 = B(0,r)$ and let $G_2 = G_1 \setminus E$. Then $G_1, G_2 \in \mathscr{O}^d$. $\chi_E(x) = \chi_{G_1 \setminus G_2}(x,y) = \chi_{G_1}(x,y) - \chi_{G_2}(x,y)$. Since $\chi_{G_1}(x,y), \chi_{G_2}(x,y) \in \mathscr{F}$ and $\chi_{G_2}(x,y) \in L(\mathbb{R}^d)$, by Lemma 4.27, $\chi_E(x,y) \in \mathscr{F}$.

(4) (E is a measure zero set) $m_d(E) = 0$. In Chapter 2, we proved that we can find a sequence open sets $\{G_n\} \subset \mathscr{O}^d$ s.t $E \subset G_n$ and $m_d(G_n) \searrow 0$ as $n \to \infty$. Without loss of generality, we may suppose that $G_{n+1} \subset G_n$ because $G_1 \cap G_2 \subset G_1$ and $G_1 \cap G_2$ is also an open set. Let $H = \bigcap_{k=1}^{\infty} G_k$ and $G_n \searrow H$. $\chi_{G_1}(x) \in L(\mathbb{R}^d), \chi_{G_k}(x,y) \searrow \chi_H(x,y)$ so by Lemma 4.27, $\chi_H(x,y) \in \mathscr{F}$.

Let $F_H(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^q} \chi_H(x,y)dy$. $\int_{\mathbb{R}^p} F_H(x)dx = \int_{\mathbb{R}^d} \chi_H(x,y)dxdy = m_d(H) = 0$. ($H$ is also a measure zero set.) From this fact, we find out that $F_H(x) = 0$ a.e $\in \mathbb{R}^p$. So for a.e $x \in \mathbb{R}^p$, $\mathscr{F}_H(x) = \int_{\mathbb{R}^p} \chi_H(x,y) = 0$. Furthermore, this implies that for a.e $x \in \mathbb{R}^p$, "$\chi_H(x,y) = 0$ a.e $y \in \mathbb{R}^q$" holds.

STEP 1. (a) Let us recall that $0 \leqq \chi_E(x,y) \leqq \chi_H(x,y)$. Therefore, for a.e $x \in \mathbb{R}^p$, "$\chi_E(x,y) = 0$ a.e $y \in \mathbb{R}^q$" also holds. So for a.e $x \in \mathbb{R}^p$, $y \mapsto \chi_E(x,y)$ is a measurable function on $\mathbb{R}^q$.

STEP 2. (b) We can define $F_E(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^q} \chi_E(x,y)dy$ a.e $x \in \mathbb{R}^p$. Though $F_E(x)$ is a function defined a.e $x \in \mathbb{R}^p$ (not defined every $x \in \mathbb{R}^p$), $F_E(x) \leqq F_H(x)$ a.e $x \in \mathbb{R}^p$, and $F_H(x) = 0$ a.e $x \in \mathbb{R}^p$ implies that $F_E(x) = 0$ a.e $x \in \mathbb{R}^p$. So we find out that $F_E(x)$ is also a measurable function on $\mathbb{R}^p$.

STEP 3. (c) Finally, $\int_{\mathbb{R}^p} F_E(x)dx = 0$ (because $F_E(x) = 0$ a.e $x \in \mathbb{R}^p$) and $\int_{\mathbb{R}^d} \chi_E(x,y)dxdy = m_d(E) = 0$. Now we conclude that $\chi_E(x,y) \in \mathscr{F}$.

(5) ($E \in \mathscr{M}$) In Chapter 2, we proved that we can decompose a measurable set $E = \bigcup_{k=1}^{\infty} F_k \cup Z$ where $\{F_k\}_{k=1}$ are bounded closed sets and Z is a measure zero set. (We may suppose $\bigcup_{k=1}^{\infty} F_k$ and Z are disjoint.) Let $K \stackrel{\text{def}}{=} \bigcup_{k=1}^{\infty} F_k$ and let $K_n \stackrel{\text{def}}{=} \bigcup_{k=1}^{n} F_k$. K_n is also a bounded closed set so $\chi_{K_n}(x,y) \in \mathscr{F}$, and $\chi_{K_n}(x,y) \nearrow \chi_K(x,y)$ so $\chi_K(x,y) \in \mathscr{F}$ by Lemma 4.27. $\chi_E(x,y) = \chi_K(x,y) + \chi_Z(x,y)$. Since both $\chi_K(x,y), \chi_Z(x,y) \in \mathscr{F}$. so $\chi_E(x,y) \in \mathscr{F}$ by Lemma 4.27.

Finally, we prove that $f(x,y) \in \mathscr{F}$ if $f(x,y)$ is a non-negative measurable function on $\mathbb{R}^d$. There exists a sequence of non-negative measurable functions $f_n(x,y) \nearrow f(x,y)$ and $f_n(x,y) \in \mathscr{F}$ so $f(x,y) \in \mathscr{F}$. ($f_n(x,y) = \sum_{i=1}^{p_n} a_{n,i} \cdot \chi_{E_{n,i}}(x,y) \in \mathscr{F}$.)

□

96 (Theorem 4.28) By Theorem 4.27, $f^+(x,y), f^-(x,y) \in \mathscr{F}$. Let $F_+(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^q} f^+(x,y)dy$ and let $F_-(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^q} f^-(x,y)dy$.

STEP 1. $(a*)$ By assumption, we have $\int_{\mathbb{R}^p} F_+(x)dx = \int_{\mathbb{R}^d} f^+(x,y)dy < \infty$ and $\int_{\mathbb{R}^p} F_-(x)dx = \int_{\mathbb{R}^d} f^-(x,y)dy < \infty$. From this fact, we have $F_+(x), F_-(x) < \infty$ a.e $x \in \mathbb{R}^p$. Furthermore, we have "$y \mapsto f^+(x,y) < \infty$ a.e $y \in \mathbb{R}^q$" a.e $x \in \mathbb{R}^p$ and "$y \mapsto f^-(x,y) < \infty$ a.e $y \in \mathbb{R}^q$" a.e $x \in \mathbb{R}^p$. Now let us fix $x \in \mathbb{R}^p$ where $y \mapsto f^+(x,y) < \infty$ a.e $y \in \mathbb{R}^q$ and $y \mapsto f^-(x,y) < \infty$ a.e $y \in \mathbb{R}^q$. Since $y \mapsto f^+(x,y) - f^-(x,y)$ is defined a.e $y \in \mathbb{R}^q$ (i.e $\infty - \infty$ does not occur a.e $y \in \mathbb{R}^q$), $y \mapsto f^+(x,y) - f^-(x,y)$ is measurable on $\mathbb{R}^q$ because sum and difference of two measurables functions are measurable as long as they are defined a.e. So for almost every $x \in \mathbb{R}^p$, "$y \mapsto f(x,y)$ is measurable on $\mathbb{R}^q$" holds.

STEP 2. $(b*)$ Let $F(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^q} f(x,y)dy$. By definition of integral, $F(x) = \int_{\mathbb{R}^q} f^+(x,y)dy - \int_{\mathbb{R}^q} f^-(x,y)dy = F_+(x) - F_-(x)$. Since $F_+(x), F_-(x) < \infty$ a.e $x \in \mathbb{R}^p$, and they are measurable on $\mathbb{R}^p$, $F(x)$ is also defined a.e $x \in \mathbb{R}^p$ (i.e $\infty - \infty$ does not occur a.e $x \in \mathbb{R}^p$), hence measurable on $\mathbb{R}^p$. (As long as $\infty - \infty$ does not occur a.e, $f - g$ is also measurable if f, g are measurable.)

STEP 3. $(c*)$ Since $\int_{\mathbb{R}^p} F_+(x)dx = \int_{\mathbb{R}^d} f^+(x,y)dxdy < \infty$ and $\int_{\mathbb{R}^p} F_-(x)dx = \int_{\mathbb{R}^d} f^-(x,y)dxdy < \infty$, we have $\int_{\mathbb{R}^p}(F_+(x) - F_-(x))dx = \int_{\mathbb{R}^d}(f^+(x,y) - f^-(x,y))dxdy$. So $\int_{\mathbb{R}^p} F(x)dx = \int_{\mathbb{R}^d} f(x,y)dxdy$. (Let us recall that if integrals of f, g exist and at least $f \in L(\mathbb{R}^d)$ or $g \in L(\mathbb{R}^d)$ holds, then $\int(f+g) = \int f + \int g$).

□

97 (Example 1)

STEP 1. Let $g(x,y) \stackrel{\text{def}}{=} \sin ax \cdot f(y) \cdot e^{-xy}$. It is easy to verify that $g(x,y)$ is a measurable function defined on $[0,\infty) \times [0,\infty)$. Let us recall that we may regard $f(y)$ is a measurable function defined on $[0,\infty) \times [0,\infty)$ because $\{ (x,y) \in [0,\infty) \times [0,\infty) \mid f(y) > t\} = [0,\infty) \times \{y \in [0,\infty) \mid f(y) > t\}$, and if $E_1, E_2 \in \mathscr{M}_1$ then $E_1 \times E_2 \in \mathscr{M}_2$.

We may regard $g(x,y)$ as a measurable function defined on $[\alpha,\beta] \times [0,\infty)$. $(0 < \alpha < \beta < \infty)$ Let us consider the following integral. Since $|g(x,y)| \leqq |f(y)| \in L([\alpha,\beta] \times [0,\infty))$, we apply Fubini's theorem $g(x,y)$ as a measurable function defined on $[\alpha,\beta] \times [0,\infty)$.

$$\begin{aligned} & \int_\alpha^\beta \left(\int_0^\infty \sin ax \cdot f(y) \cdot e^{-xy} dy \right) dx \\ = & \int_0^\infty \left(\int_\alpha^\beta \sin ax \cdot f(y) \cdot e^{-xy} dx \right) dy \end{aligned}$$

STEP 2. Let us define

$$G_{\alpha,\beta}(y) \stackrel{\text{def}}{=} \int_\alpha^\beta \sin ax \cdot f(y) \cdot e^{-xy} dx.$$

We prove that $G_{\alpha,\beta}(y)$ is bounded by an integrable function. (We would like to use Lebesgue Dominated Convergence Theorem later.) Since

$$\begin{aligned} & (\mathrm{R}, \mathrm{L}) \int_\alpha^\beta \sin ax e^{-xy} dx \\ = & \frac{1}{y^2 + a^2} \left(y \sin a\alpha e^{-\alpha y} + a \cos a\alpha e^{-\alpha y} - y \sin a\beta e^{-\beta y} - a \cos a\beta e^{-\beta y} \right), \end{aligned}$$

and by triangular inequality,

$$\left|\int_\alpha^\beta \sin ax e^{-xy} dx\right| \leqq \frac{2y+2a}{y^2+a^2} \overset{*1}{\leqq} 2,$$

we have

$$|G_{\alpha,\beta}(y)| \leqq 2\,|f(y)| \in L([0,\infty)).$$

- $(*1)$ $\frac{2y+2a}{y^2+a^2} = \frac{2+2t}{1+t^2}$ where $t = \frac{a}{y} > 0$.

STEP 3. Finally, by Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}
&\lim_{\alpha\to+0,\beta\to\infty} \int_\alpha^\beta \left(\int_0^\infty \sin ax \cdot f(y) \cdot e^{-xy} dy\right) dx \\
\overset{*2}{=}\ &\lim_{\alpha\to+0,\beta\to\infty} \int_0^\infty \left(\int_\alpha^\beta \sin ax \cdot f(y) \cdot e^{-xy} dx\right) dy \\
\overset{*3}{=}\ &\int_0^\infty \left(\lim_{\alpha\to+0,\beta\to\infty} \int_\alpha^\beta \sin ax \cdot f(y) \cdot e^{-xy} dx\right) dy \\
\overset{*4}{=}\ &\int_0^\infty \left(\frac{af(y)}{y^2+a^2}\right) dy
\end{aligned}$$

- $(*2)$ Step1
- $(*3)$ Lebesgue Dominated Convergence theorem
- $(*4)$ $\lim_{\alpha\to+0,\beta\to\infty} f(y)\cdot\frac{1}{y^2+a^2}\left(y\sin a\alpha e^{-\alpha y} + a\cos a\alpha e^{-\alpha y} - y\sin a\beta e^{-\beta y} - a\cos a\beta e^{-\beta y}\right)$

□

98 (Example 2) Let us consider the following integral and apply Tonelli's Theorem.

$$\begin{aligned}
&\int_{x,y\in[0,\infty)\times[0,\infty)} 2y\exp\left(-(1+x^2)y^2\right) dxdy \\
=\ &\int_{x\in[0,\infty)} \left(\int_{y\in[0,\infty)} 2y\exp\left(-(1+x^2)y^2\right) dy\right) dx \ \cdots\ (i) \\
=\ &\int_{y\in[0,\infty)} \left(\int_{x\in[0,\infty)} 2y\exp\left(-(1+x^2)y^2\right) dx\right) dy \ \cdots\ (ii)
\end{aligned}$$

STEP 1. First we find (i). $\int_{y\in[0,\infty)} 2y\exp(-(1+x^2)y^2)dy = \frac{1}{1+x^2}$ because by monotone convergence theorem,

$$\lim_{c\to\infty} (\mathrm{L}) \int_{y\in[0,c]} 2y\exp(-(1+x^2)y^2)dy.$$

Moreover $2y\exp(-(1+x^2)y^2)$ is Riemann integrable on $[0,c]$ so we find

$$\lim_{c\to\infty} (\mathrm{R}) \int_{y\in[0,c]} 2y\exp(-(1+x^2)y^2)dy.$$

And this is $\frac{1}{1+x^2}$. Finally, we find $\int_{x\in[0,\infty)} \frac{1}{1+x^2}dx$. Similarly, we can find the integral as Riemann improper integral and we have $\int_{x\in[0,\infty)} \frac{1}{1+x^2} = \frac{\pi}{2}$.

STEP 2. Second we find (ii).

$$\begin{aligned}
&\int_{y\in[0,\infty)}\left(\int_{x\in[0,\infty)} 2y\exp\left(-(1+x^2)y^2\right)dx\right)dy\\
=&\int_{y\in[0,\infty)} 2y\exp(-y^2)\left(\int_{x\in[0,\infty)}\exp\left(-x^2y^2\right)dx\right)dy
\end{aligned}$$

We consider

$$\int_{x\in[0,\infty)}\exp\left(-x^2y^2\right)dx.$$

Since $m(\{0\})=0$ and by monotone convergence theorem,

$$=\lim_{c\to\infty}\int_{x\in(0,c)}\exp\left(-x^2y^2\right)dx.$$

We apply §4.2 Example 10 (let $z\stackrel{\text{def}}{=}yx$),

$$=\lim_{c\to\infty}\frac{1}{y}\int_{z\in(0,yc)}\exp\left(-z^2\right)dz.$$

Again by monotone convergence theorem, we have

$$\begin{aligned}
&=\frac{1}{y}\int_{z\in(0,\infty)}\exp\left(-z^2\right)dz\\
&=\frac{1}{y}\int_{z\in[0,\infty)}\exp\left(-z^2\right)dz
\end{aligned}$$

Therefore

$$(ii)\ =2\cdot\left(\int_{[0,\infty)}\exp(-y^2)dy\right)^2.$$

Finally, $2\cdot\left(\int_{[0,\infty)}\exp(-y^2)dy\right)^2=\frac{\pi}{2}$ and we have $\int_{[0,\infty)}\exp(-y^2)dy=\frac{\sqrt{\pi}}{2}$.

□

99 (Exercise 1) Since $f(x,y)$ is integrable on $[0,1]\times[0,1]$, we apply Fubini's Theorem.

$$\begin{aligned}
\int_0^1\left(\int_0^x f(x,y)dy\right)dx &= \int_0^1\left(\int_0^1 f(x,y)\cdot\chi_{[0,x]}(y)dy\right)dx\\
&\stackrel{*1}{=} \int_0^1\left(\int_0^1 f(x,y)\cdot\chi_{[0,x]}(y)dx\right)dy\\
&\stackrel{*2}{=} \int_0^1\left(\int_0^1 f(x,y)\cdot\chi_{[y,1]}(x)dx\right)dy\\
&= \int_0^1\left(\int_y^1 f(x,y)dx\right)dy
\end{aligned}$$

- (∗1) Fubini's Theorem
- (∗2) When $0 \leqq x, y \leqq 1$, $0 \leqq y \leqq x$ if and only if $y \leqq x \leqq 1$

□

100 (Exercise 2) We apply Tonell's Theorem.

$$\begin{aligned}
\int_{\mathbb{R}^d} m(A_{-x} \cap B)dx &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \chi_{A_{-x} \cap B}(y)dy \right) dx \\
&\overset{*1}{=} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \chi_{A_{-x}}(y) \cdot \chi_B(y)dy \right) dx \\
&\overset{*2}{=} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \chi_A(x+y) \cdot \chi_B(y)dy \right) dx \\
&\overset{*3}{=} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \chi_A(x+y) \cdot \chi_B(y)dx \right) dy \\
&\overset{*4}{=} \int_{\mathbb{R}^d} \chi_B(y) \cdot \left(\int_{\mathbb{R}^d} \chi_A(x+y)dx \right) dy \\
&\overset{*5}{=} \int_{\mathbb{R}^d} \chi_B(y) \cdot \left(\int_{\mathbb{R}^d} \chi_A(x)dx \right) dy \\
&= \int_{\mathbb{R}^d} \chi_B(y) \cdot m(A)dy \\
&= m(A) \cdot \int_{\mathbb{R}^d} \chi_B(y)dy \\
&= m(A) \cdot m(B)
\end{aligned}$$

- (∗1) $\chi_{A_1}(x)\chi_{A_2}(x) = \chi_{A_1 \cap A_2}(x)$
- (∗2) $y \in A_{-x}$ if and only if $x + y \in A$
- (∗3) Tonelli's Theorem
- (∗4) $\chi_B(y)$ is not related to x so we may put it outside of $\int_{\mathbb{R}^d} \cdots\ dx$ by linearity of integral.
- (∗5) Theorem 4.13

□

101 (Theorem 4.30) Let $E \in \mathscr{M}_d$ where $d = p + q$. Let us consider a measurable function $\chi_E(x, y), x \in \mathbb{R}^p, y \in R^q$. We apply Tonell's Theorem to $\chi_E(x, y)$. For a.e $x \in \mathbb{R}^p$, $y \mapsto \chi_E(x, y)$ is a measurable function on $\mathbb{R}^q$. When $x \in \mathbb{R}^p$ is fixed, $\chi_E(x, y) = \chi_{E_{|x}}(y)$. So for a.e $x \in \mathbb{R}^p$, $y \mapsto \chi_{E_{|x}(y)}$ is a measurable function on R^q.

Furthermore,

$$\int_{\mathbb{R}^d} \chi_E(x, y)dxdy = \int_{\mathbb{R}^p} \int_{\mathbb{R}^q} \chi_E(x, y)dydx.$$

The right hand side is

$$\int_{\mathbb{R}^p}\int_{\mathbb{R}^q}\chi_{E_{|x}(y)}dydx$$
$$= \int_{\mathbb{R}^p} m(E_{|x})dx.$$

Now we have the desired conclusion. □

102 (Theorem 4.31)

(1) We show that $E_1 \in \mathscr{M}_p, E_2 \in \mathscr{M}_q$ then $E_1 \times E_2 \stackrel{\text{def}}{=} \{(x,y) \mid x \in E_1, y \in E_2\} \in \mathscr{M}_{p+q}$. Let us recall that $E_1 = K_1 \cup Z_1, E_2 = K_2 \cup Z_2$ where K_1, K_2 are F_σ sets and Z_1, Z_2 are measure zero sets. So $E_1 \times E_2 = K_1 \times K_2 \cup K_1 \times Z_2 \cup Z_1 \times K_2 \cup Z_1 \times Z_2$.

STEP 1. First we show that $K_1 \times K_2$ is measurable on $\mathbb{R}^{p+q}$. $K_1 \times K_2 = \bigcup_{i\in\mathbb{N}} F_{1,i} \times \bigcup_{j\in\mathbb{N}} F_{2,j}$ where $F_{1,i}, F_{2,j}$ are closed sets on $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. So we prove that $\bigcup_{i,j\in\mathbb{N}} F_{1,i} \times F_{2,j}$ is measurable. It is enough for us to prove that $F_1 \times F_2$ (F_1, F_2 are closed sets.) is also a closed set. Let $\{(x_n, y_n)\} \subset F_1 \times F_2$ and $(x_n, y_n) \to (x, y)$. Since $x_n \to x \in F_1$ and $y_n \to y \in F_2$, $(x,y) \in F_1 \times F_2$. So $F_1 \times F_2$ is also a closed set on $\mathbb{R}^{p+q}$.

STEP 2. We prove that $A \times Z$ is a measure zero set if $m_p^*(A) < \infty$ and $m_q(Z) = 0$ ($A \subset \mathbb{R}^p, B \subset \mathbb{R}^q$). Let $a \stackrel{\text{def}}{=} m_p^*(A) < \infty$. We can find open intervals $\{I_{1,n}\}$ on $\mathbb{R}^p$ s.t $A \subset \bigcup_{n\in\mathbb{N}} I_{1,n}$ and $a \leqq \sum_{n=1}^\infty |I_{1,n}| < a+1$. Let $\epsilon > 0$ be an arbitrary positive number. We can also find open intervals $\{I_{2,m}\}$ on $\mathbb{R}^q$ s.t $Z \subset \bigcup_{m\in\mathbb{N}} I_{2,m}$ and $\sum_{m=1}^\infty |I_{2,m}| < \epsilon$. Then $A \times Z \subset \bigcup_{n\in\mathbb{N}} I_{1,n} \times \bigcup_{m\in\mathbb{N}} I_{2,m} = \bigcup_{n,m\in\mathbb{N}} I_{1,n} \times I_{2,m}$. So $m_{p+q}^*(A \times Z) \leqq \sum_{n=1}^\infty \sum_{m=1}^\infty |I_{1,n}||I_{2,m}| = \sum_{n=1}^\infty |I_{1,n}| \sum_{m=1}^\infty |I_{2,m}| < (1+a)\cdot\epsilon$.

STEP 3. In Step2, we assumed that $m_p^*(A) < \infty$, however $A \times Z$ is still a measure zero set when $m_p^*(A) = \infty$. Let $A_n \stackrel{\text{def}}{=} A \cap B(0,n)$ where $B(0,n)$ is an open ball with radius n whose center is at origin. Then $A = \bigcup_{n\in\mathbb{N}} A_n$. So $A \times Z = \bigcup_{n\in\mathbb{N}} A_n \times Z$. Each $A_n \times Z$ is a measure zero set so $A \times Z$ is also a measure zero set. (A_n is bounded so it has a finite measure.)

From the arguments above, we find out that $E_1 \times E_2$ is measurable on $\mathbb{R}^{p+q}$.

(2) We apply Tonell's Theorem to $\chi_{E_1\times E_2}(x,y) = \chi_{E_1}(x)\cdot\chi_{E_2}(y)$. (This equality holds obviously.) Let us consider $\int_{\mathbb{R}^{p+q}} \chi_{E_1\times E_2}(x,y)dxdy$ and $\int_{\mathbb{R}^p}\left(\int_{\mathbb{R}^q}\chi_{E_1}(x)\cdot\chi_{E_2}(y)dy\right)dx$. The left hand side is $m_{p+q}(E_1 \times E_2)$, and the right hand side is $m_p(E_1) \times m_q(E_2)$. Now we have the desired conclusion.

□

103 (Corollary 4.32) Let $E_k \stackrel{\text{def}}{=} \{x \in E \mid (k-1)\cdot\delta \leqq f(x) < k\cdot\delta\}$. Without loss of generality, we may suppose that $m_d(E) < \infty$.

STEP 1. Let $\delta > 0$ be an arbitrary positive number.

$$G(E;f) = \bigcup_{k=1}^\infty G(E_k;f).$$

Since $f(x)$ is real-valued, $E = \bigcup_{k=1}^{\infty} E_k$. Therefore $\{(x,y) \in \mathbb{R}^{d+1} \mid x \in E, y = f(x)\} = \bigcup_{k=1}^{\infty}\{(x,y) \in \mathbb{R}^{d+1} \mid x \in E_k, y = f(x)\}$. (The left hand side is $G(E;f)$ and the right hand side is $\bigcup_{k=1}^{\infty} G(E_k;f)$)

STEP 2. By sub-additivity of an outer measure ($m^*_{d+1}(\cdot)$),

$$m^*_{d+1}(G(E;f)) \leqq \sum_{k=1}^{\infty} m^*_{d+1}(G(E_k;f)).$$

Moreover, $G(E_k;f) \subset E_k \times \{y \in \mathbb{R} \mid (k-1)\cdot\delta \leqq y < k\cdot\delta\}$, so

$$m^*_{d+1}(G(E_k;f)) \leqq m_{d+1}(E_k \times [(k-1)\cdot\delta, k\cdot\delta)) = m_d(E_k)\cdot\delta.$$

(In the inequality above, since $E_k \in \mathscr{M}_d$ and $[(k-1)\cdot\delta, k\cdot\delta) \in \mathscr{M}$, $E_k \times [(k-1)\cdot\delta, k\cdot\delta) \in \mathscr{M}_{d+1}$). Therefore we have

$$m^*_{d+1}(G(E;f)) \leqq \sum_{k=1}^{\infty} m^*_{d+1}(G(E_k;f)) \leqq \sum_{k=1}^{\infty} \delta\cdot m_d(E_k) = \delta\cdot m_d(E).$$

Since $m_d(E) < \infty$, by taking $\delta \searrow 0$, we have the desired conclusion.

STEP 3. If $m(E) = \infty$, we consider $E_r \overset{\text{def}}{=} E \cap B(0,r), r = 1,2,3\cdots$. Then $E = \bigcup_{r=1}^{\infty} E_r$ hence $G(E,f) = \bigcup_{r=1}^{\infty} G(E_r,f)$. $m_{d+1}(G(E_r;f)) = 0$ for each $r = 1,2,3\cdots$. So $m_{d+1}(\bigcup_{r=1}^{\infty} G(E_r;f)) = 0$.

□

104 (Theorem 4.33 -1)

STEP 1. ($f(x)$ is a non-negative measurable simple function.) Suppose that $f(x) \overset{\text{def}}{=} \sum_{i=1}^{p} a_i\chi_{A_i}(x), A_i \in \mathscr{M}, A_i \subset E$. Suppose that $A_1,\cdots A_p$ are disjoint and $E = \bigcup_{i=1}^{p} A_i$. Then $\underline{G}(E;f) = \{(x,y) \in \mathbb{R}^{d+1} \mid x \in E, 0 \leqq y \leqq \sum_{i=1}^{p} a_i\chi_{A_i}(x)\}$. When $x \in A_i$, $0 \leqq y \leqq a_i$. Therefore $\underline{G}(E;f) = \bigcup_{i=1}^{p}\{(x,y) \in \mathbb{R}^{d+1} \mid x \in A_i, 0 \leqq y \leqq f(x)\} = \bigcup_{i=1}^{p} A_i \times [0,a_i]$ (this is a disjoint union). So we have $m_{d+1}(\underline{G}(E;f)) = m_{d+1}\left(\bigcup_{i=1}^{p} A_i \times [0,a_i]\right) = \sum_{i=1}^{p} a_i m_d(A_i) = \int_E f(x)dx$.

STEP 2. ($f(x)$ is a non-negative measurable function.) We find a sequence of non-negative measurable simple functions $f_n(x) \nearrow f(x)$. By monotone convergence theorem, we have

$$\lim_{n\to\infty}\int_E f_n(x)dx = \int_E f(x)dx.$$

The left hand side is

$$\lim_{n\to\infty}\int_E f_n(x)dx = \lim_{n\to\infty} m_{d+1}(\underline{G}(E;f_n)).$$

Since $f_n \leqq f_{n+1}$, $\underline{G}(E;f_n) \subset \underline{G}(E;f_{n+1})$. Therefore the right hand side is

$$\lim_{n\to\infty} m_{d+1}(\underline{G}(E;f_n)) = m_{d+1}\left(\bigcup_{n=1}^{\infty} \underline{G}(E;f_n)\right).$$

Let us consider $\bigcup_{n=1}^{\infty} \underline{G}(E; f_n)$.

$$\begin{aligned} & \{(x,y) \in \mathbb{R}^{d+1} \mid x \in E, 0 \leqq y < f(x)\} \\ \overset{*1}{\subset} & \bigcup_{n=1}^{\infty} \underline{G}(E; f_n) = \bigcup_{n=1}^{\infty} \{(x,y) \in \mathbb{R}^{d+1} \mid x \in E, 0 \leqq y \leqq f_n(x)\} \\ \overset{*2}{\subset} & \{(x,y) \in \mathbb{R}^{d+1} \mid x \in E, 0 \leqq y \leqq f(x)\} = \underline{G}(E; f). \end{aligned}$$

- ($*1$) Equality does not necessarily hold. However $\bigcup_{n=1}^{\infty} \{(x,y) \in \mathbb{R}^{d+1} \mid x \in E, 0 \leqq y < f_n(x)\} = \{(x,y) \in \mathbb{R}^{d+1} \mid x \in E, 0 \leqq y < f(x)\}$.
- ($*2$) Equality does not necessarily hold. If for all $x \in E$, there exists $N \in \mathbb{N}$, $f_n(x) = f(x), \forall n \geqq N$ then the equality hold.

Therefore, we have

$$\bigcup_{n=1}^{\infty} \underline{G}(E; f_n) \cup G(E; f) = \underline{G}(E; f).$$

Since $m_{d+1}(G(E; f)) = 0$, we have

$$m_{d+1}\left(\bigcup_{n=1}^{\infty} \underline{G}(E; f_n) \right) = m_{d+1}\left(\underline{G}(E; f) \right).$$

The left hand side is $\int_E f(x)dx$. Now the proof is complete.

□

105 (Theorem 4.33 -2) Let us consider $\underline{G}(E; f)|_{y=a}$. $\underline{G}(E; f)|_{y=a} = \{x \in E \mid f(x) \geqq a\}$. (If you do not know why, you may draw a graph.) By Tonelli's Theorem, $x \mapsto \chi_{\underline{G}(E;f)}$ is a measurable function for a.e $y \in \mathbb{R}$. Therefore, $\underline{G}(E; f)|_y = \{x \in E \mid f(x) \geqq y\}$ is Lebesgue measurable for a.e $y \in \mathbb{R}$. ($*$) Let $t \in \mathbb{R}$ be an arbitrary real number. We can find a sequence of $\{y_k\}_{k \geqq 1}$ s.t $y_k \searrow t$ and $\{x \in E \mid f(x) \geqq y_k\}$ is measurable for all $k \in \mathbb{N}$. (Otherwise, there exists an interval $(c,d) \subset \mathbb{R}$ s.t $\forall y \in (c,d)$, $\{x \in E \mid f(x) \geqq y\} \notin \mathscr{M}_d$. This contradicts to ($*$).) So $\{x \in E \mid f(x) > t\} = \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geqq y_k\} \in \mathscr{M}_d$.

□

106 (Definition of Convolution) If $f(x), g(x)$ are measurable functions on $\mathbb{R}^d$ and $f(x-y)g(y)$ is integrable with respect to y then we define the convolution of $f(x)$ and $g(x)$ as

$$(f * g)(x) \overset{\text{def}}{=} \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

□

107 (Theorem 4.34)

STEP 1. Let us recall Corollary 3.24. $(x,y) \mapsto f(x-y)$ is a Lebesgue measurable function on $\mathbb{R}^{2d}$. $(x,y) \mapsto g(y)$ is also a Lebesgue measurable function on $\mathbb{R}^{2d}$. Therefore $f(x-y)g(y)$ is a Lebesgue measurable function on $\mathbb{R}^{2d}$.

STEP 2. By Tonelli's Theorem

$$\begin{aligned}\int_{\mathbb{R}^{2d}} |f(x-y)g(y)|\, dxdy &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y)g(y)|\, dx \right) dy \\ &= \int_{\mathbb{R}^d} |g(y)| \left(\int_{\mathbb{R}^d} |f(x-y)|\, dx \right) dy \\ &\overset{*}{=} \int_{\mathbb{R}^d} |g(y)| \left(\int_{\mathbb{R}^d} |f(x)|\, dx \right) dy \\ &= \int_{\mathbb{R}^d} |g(y)|\, dy \cdot \int_{\mathbb{R}^d} |f(x)|\, dx < \infty\end{aligned}$$

.

- $(*)$ Theorem 4.13.

Therefore $f(x-y)g(y) \in L(\mathbb{R}^{2d})$. By Fubini's Theorem,

$$(f * g)(x) \overset{\text{def}}{=} \int_{\mathbb{R}^d} f(x-y)g(y)dy$$

is a measurable function on $\mathbb{R}^d$.

□

108 (Example 5) Suppose there exists $u(x) \in L(\mathbb{R})$ s.t $\forall f(x) \in L(\mathbb{R})$,

$$f(x) = (u * f)(x) \text{ a.e } x \in \mathbb{R}.$$

STEP 1. Let us apply Lebesgue Dominated Convergence Theorem to $u(x) \cdot \chi_{[-2\delta, 2\delta]}(x)$. Since $|u(x)| \cdot \chi_{[-2\delta, 2\delta]}(x) \leqq |u(x)| \in L(\mathbb{R}^1)$, by taking $\delta \searrow 0$, we have

$$\lim_{\delta \to +0} \int_{\mathbb{R}} |u(x)| \cdot \chi_{[-2\delta, 2\delta]}(x) dx = 0.$$

This implies that there exists sufficiently small $\delta > 0$ s.t

$$\int_{\mathbb{R}} |u(x)| \cdot \chi_{[-2\delta, 2\delta]}(x) dx < 1.$$

So $\int_{[-2\delta, 2\delta]} |u(x)|\, dx < 1$.

STEP 2. Let $f(x) \stackrel{\text{def}}{=} \chi_{[-\delta,\delta]}$. By assumption,

$$
\begin{aligned}
f(x) \stackrel{\text{a.e } x\in\mathbb{R}}{=} (u * f)(x)dx &= \int_{\mathbb{R}} u(x-y)f(y)dy \\
&= \int_{\mathbb{R}} u(x-y)\chi_{[-\delta,\delta]}(y)dy \\
&\stackrel{*1}{=} \int_{\mathbb{R}} u(-y)\chi_{[-\delta,\delta]}(y+x)dy \\
&= \int_{\mathbb{R}} u(-y)\chi_{[-\delta-x,\delta-x]}(y)dy \\
&= \int_{[-\delta-x,\delta-x]} u(-y)dy \\
&\stackrel{*2}{=} \int_{[x-\delta,x+\delta]} u(y)dy
\end{aligned}
$$

- (∗1) Theorem 4.13. Translation does not change the value of integral. $y \to y + x$
- (∗2) See §4.2. Example 10. $a = -1$.

So there exists $x_0 \in [-\delta, \delta]$ s.t

$$
f(x_0) = 1 \quad = \quad \int_{[x_0-\delta,x_0+\delta]} u(y)dy
$$

STEP 3. Finally, consider

$$
\begin{aligned}
1 = \left| \int_{[x_0-\delta,x_0+\delta]} u(y)dy \right| &\leqq \int_{[x_0-\delta,x_0+\delta]} |u(y)|\, dy \\
&\stackrel{*3}{\leqq} \int_{[-2\delta,2\delta]} |u(y)|\, dy < 1.
\end{aligned}
$$

- (∗3) Since $x_0 \in [-\delta, \delta]$, $-2\delta \leqq x_0 - \delta \leqq x_0 + \delta \leqq 2\delta$

So there is a contradiction.

□

109 (Definition 4.4) Let $f(x)$ be a measurable function defined on $E \in \mathscr{M}$. We define the distribution function $f_*(\lambda)$ as

$$
f_*(\lambda) \stackrel{\text{def}}{=} m\left(\{x \in E \mid |f(x)| > \lambda\}\right)
$$

□

110 (Theorem 4.35) We use Tonell's Theorem.

$$\begin{aligned}
\int_E |f(x)|^p\,dx &\overset{*1}{=} \int_E \left((\mathrm{R})\int_{[0,f(x)]} p\lambda^{p-1}d\lambda\right)dx \\
&\overset{*2}{=} \int_E \left((\mathrm{L})\int_{[0,f(x)]} p\lambda^{p-1}d\lambda\right)dx \\
&\overset{*3}{=} \int_E \left(\int_{[0,f(x))} p\lambda^{p-1}d\lambda\right)dx \\
&= \int_E \left(\int_{[0,\infty)} p\lambda^{p-1}\chi_{[0,f(x))}(\lambda)d\lambda\right)dx \\
&= \int_E \left(\int_{[0,\infty)} p\lambda^{p-1}\chi_{\{x\in E|f(x)>\lambda\}}(x)d\lambda\right)dx \\
&\overset{*4}{=} \int_{[0,\infty)} \left(\int_E p\lambda^{p-1}\chi_{\{x\in E|f(x)>\lambda\}}(x)dx\right)d\lambda \\
&= \int_{[0,\infty)} p\lambda^{p-1}\cdot\left(\int_E \chi_{\{x\in E|f(x)>\lambda\}}(x)dx\right)d\lambda \\
&= \int_{[0,\infty)} p\lambda^{p-1}\cdot m(\{x\in E\mid f(x)>\lambda\})d\lambda \\
&= \int_{[0,\infty)} p\lambda^{p-1}\cdot f_*(\lambda)d\lambda
\end{aligned}$$

- $(*1)$ (R) $\int_{[0,a]} pt^{p-1}dt = a^p$.
- $(*2)$ Riemann integrable implies Lebesgue integrable. (The integrals are all Lebesgue integrals from the second line.)
- $(*3)$ a single point is a measure zero set. So the integral does not change even if we get rid of it from the range of integral.
- $(*4)$ Tonell's Theorem. (When the function is non-negative, we may always swap the order of iterated integrals.)

□

§ 4.6

111 (Exercise 1)

$$\begin{aligned}
0=\int_E f(x)dx &\geqq \int_{\{x\in E|f(x)>1/n\}} f(x)dx \\
&\geqq \int_{\{x\in E|f(x)>1/n\}} \frac{1}{n}dx \\
&\geqq \frac{1}{n}m(\{x\in E\mid f(x)>1/n\})
\end{aligned}$$

So $m(\{x \in E \mid f(x) > 1/n\}) = 0$ for all $n \in \mathbb{N}$. Therefore

$$m\left(\bigcup_{n=1}^{\infty}\{x \in E \mid f(x) > 1/n\}\right) = m(\{x \in E \mid f(x) > 0\}) = m(E) = 0$$

□

112 (Exercise 2) Let $\epsilon > 0$ be a positive number. Since $f'(0)$ exists, $\exists \delta > 0$ s.t $\left|\frac{f(x)}{x} - f'(0)\right| < \epsilon$ for all $x \in (0, \delta)$. And we have $\left|\frac{f(x)}{x}\right| \leqq \left|\frac{f(x)}{x} - f'(0)\right| + |f'(0)| = M < \infty$ for all $x \in (0, \delta)$. Since $f(x)$ is non-negative, $\frac{f(x)}{x} \leqq M$.

$$\begin{aligned}
\int_{(0,\infty)} \frac{f(x)}{x}dx &= \int_{(0,\delta)} \frac{f(x)}{x}dx + \int_{[\delta,\infty)} \frac{f(x)}{x}dx \\
&\leqq \int_{(0,\delta)} M dx + \int_{[\delta,\infty)} \frac{f(x)}{x}dx \\
&\leqq \int_{(0,\delta)} M dx + \int_{[\delta,\infty)} \frac{f(x)}{\delta}dx \\
&= M \cdot \delta + \frac{1}{\delta}\int_{[\delta,\infty)} f(x)dx \\
&\leqq M \cdot \delta + \frac{1}{\delta}\int_{(0,\infty)} f(x)dx < \infty
\end{aligned}$$

□

113 (Exercise 3) First we show some fundamental facts.

STEP 1. Let $\{a_n\}, \{b_n\}$ be sequences of real-numbers. We show that $\liminf_{n\to\infty}(a_n + b_n) \leqq \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$. (We suppose that both limits on the right hand side are finite.)

$$\begin{aligned}
\liminf_{n\to\infty}(a_n + b_n) &= \lim_{n\to\infty}\inf_{m\geqq n}(a_m + b_m) \\
&\overset{*1}{\leqq} \lim_{n\to\infty}\inf_{m\geqq n}(a_m + \sup_{m'\geqq n} b_{m'}) \\
&\overset{*2}{\leqq} \lim_{n\to\infty}(\inf_{m\geqq n} a_m + \sup_{m'\geqq n} b_{m'}) \\
&\overset{*3}{=} \lim_{n\to\infty}\inf_{m\geqq n} a_m + \lim_{n\to\infty}\sup_{m'\geqq n} b_{m'}
\end{aligned}$$

- ($*1$) when $m \geqq n$, $b_m \leqq \sup_{m'\geqq n} b_{m'}$
- ($*2$) $\sup_{m'\geqq n} b_{m'}$ is not related to m and finite for sufficiently large n, so we can separate.
- ($*3$) $\lim_{n\to\infty}(c_n + d_n) = \lim_{n\to\infty} c_n + \lim_{n\to\infty} d_n$ when both c_n, d_n converge.

STEP 2. Let $E_k \in \mathscr{M}$ for all $k \geqq 1$. We show that

$$A \stackrel{\text{def}}{=} \limsup_{k\to\infty} \chi_{E_k}(x) = B \stackrel{\text{def}}{=} \chi_{\limsup_{k\to\infty} E_k}(x).$$

We show $A \leqq B$ and $A \geqq B$. A, B only take 0 or 1. It is enough for us to show that $A = 1 \Rightarrow B = 1$ and $B = 1 \Rightarrow A = 1$.

Let us consider that x is fixed. First if $A = 1$, there are infinitely many $k \in \mathbb{N}$ s.t $\chi_{E_k}(x) = 1$. In other words, there are infinitely many $k \in \mathbb{N}$ s.t $x \in E_k$. So $x \in \limsup_{k\to\infty} E_k$. Hence $B = 1$.

Next, if $B = 1$, then $x \in \limsup_{k\to\infty} E_k$. This means that x is contained infinitely many $k \in \mathbb{N}$. So $\chi_{E_k}(x) = 1$ occurs infinitely many times. Hence $\limsup_{k\to\infty} \chi_{E_k(x)} \geqq 1$. But $\chi_{E_k}(x) \leqq 1$. Therefore $A = \limsup_{k\to\infty} \chi_{E_k(x)} = 1$.

STEP 3. Let $A_k \stackrel{\text{def}}{=} E \setminus E_{2^k}$ and $B_k \stackrel{\text{def}}{=} E_{2^k}$. Then $m(A_k) < \frac{1}{2^k}$. This implies that $m(\limsup_{k\to\infty} A_k) = 0$ by Borel-Cantelli's lemma. (See §2.2. Example 2)

$$\begin{aligned}
\int_E f(x)dx &= \int_E (f(x)\cdot\chi_{A_k}(x) + f(x)\cdot\chi_{B_k}(x))dx \\
&\stackrel{*1}{=} \int_E \liminf_{k\to\infty}(f(x)\cdot\chi_{A_k}(x) + f(x)\cdot\chi_{B_k}(x))dx \\
&\stackrel{*2}{\leqq} \int_E (\limsup_{k\to\infty} f(x)\cdot\chi_{A_k}(x) + \liminf_{k\to\infty} f(x)\cdot\chi_{B_k}(x))dx \\
&\stackrel{*3}{\leqq} \int_E (f(x)\cdot\chi_{\limsup_{k\to\infty} A_k}(x) + \liminf_{k\to\infty} f(x)\cdot\chi_{B_k}(x))dx \\
&\stackrel{*4}{=} \int_E \liminf_{k\to\infty} f(x)\cdot\chi_{B_k}(x)dx \\
&\stackrel{*5}{\leqq} \liminf_{k\to\infty} \int_E f(x)\cdot\chi_{B_k}(x)dx \\
&\stackrel{*6}{=} \liminf_{k\to\infty} \int_{B_k} f(x)dx \\
&\stackrel{*7}{=} \liminf_{k\to\infty} \int_{E_{2^k}} f(x)dx \\
&\stackrel{*8}{=} \lim_{k\to\infty} \int_{E_{2^k}} f(x)dx < \infty
\end{aligned}$$

- (∗1) $f(x)\cdot\chi_{A_k}(x) + f(x)\cdot\chi_{B_k}(x)$ is not related to k. It does not change even if we take $\liminf_{k\to\infty}$.
- (∗2) We apply the fact stated in Step 1.
- (∗3) We apply the fact stated in Step 2.
- (∗4) $m(\limsup_{k\to\infty} A_k) = 0$. So the first term in $\int_E(\cdots)$ equals to 0 a.e $x \in E$.
- (∗5) Fatou's lemma.

- (*6) This is a basic property about Lebesgue integral of non-negative measurable functions.
- (*7) Let us recall that $B_k \stackrel{\text{def}}{=} E_{2^k}$.
- (*8) By assumption, the limit exists. So we may change lim inf to lim. And it converges. So it is finite.

□

114 (Exercise 4) We use Tonell's Theorem.

$$\begin{aligned}
\infty > \int_{\mathbb{R}} F(x)dx &= \int_{\mathbb{R}} \int_{(-\infty,x]} f(t)dtdx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) \cdot \chi_{(-\infty,x]}(t)dtdx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) \cdot \chi_{[t,\infty)}(x)dtdx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) \cdot \chi_{[t,\infty)}(x)dxdt \\
&= \int_{\mathbb{R}} f(t) \int_{\mathbb{R}} \chi_{[t,\infty)}(x)dxdt \\
&= \int_{\mathbb{R}} f(t) \cdot m([t,\infty))dt \\
&= \int_{\mathbb{R}} \infty \cdot f(t)dt \\
&= \infty \cdot \int_{\mathbb{R}} f(t)dt
\end{aligned}$$

So we conclude that $\int_{\mathbb{R}} f(t)dt = 0$. (In Lebesgue Integral, $0 \cdot \infty = 0$) □

115 (Exercise 5) Let $E_k \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid f_k(x) > f_{k+1}(x)\}$. Since $\{f_k(x)\}_{k \geqq 1}$ are integrable, $\int_{E_k} f_{k+1}(x)dx - \int_{E_k} f_k(x)dx = \int_{E_k}(f_{k+1}(x) - f_k(x))dx \geqq 0$. $f_{k+1}(x) - f_k(x) < 0$ on $x \in E_k$, so $m(E_k) = 0$, otherwise $\int_{E_k}(f_{k+1}(x) - f_k(x))dx < 0$. And we have $m(\bigcup_{k=1}^{\infty} E_k) = 0$. This implies that $f_1(x) \leqq f_2(x) \leqq \cdots \leqq f_k(x) \leqq \cdots$ a.e $x \in \mathbb{R}^d$. Finally, we have the desired conclusion by monotone convergence theorem. □

116 (Exercise 6)

STEP 1. Since $(\sqrt{f(x)} - \sqrt{g(x)})^2 = f(x) - 2\sqrt{f(x)g(x)} + g(x) \geqq 0$, we have $f(x) + g(x) \geqq 2\sqrt{f(x)g(x)} \geqq 0$. So $\sqrt{f(x)g(x)}$ is also integrable on E.

STEP 2. Let us consider an equation with respect to t, $\int_E (t \cdot \sqrt{f(x)} - \sqrt{g(x)})^2 dx = 0$. (This equation has at most one root.) $\int_E (t \cdot \sqrt{f(x)} - \sqrt{g(x)})^2 dx = \int_E (t^2 f(x) - 2t\sqrt{f(x)g(x)} + g(x))dx$. Since $f(x), g(x), \sqrt{f(x)g(x)}$ are all integrable on E, we have $= t^2 \cdot \int_E f(x)dx - t \cdot 2\int_E \sqrt{f(x)g(x)}dx + \int_E g(x)dx$. The discriminant of the quadratic equation is equal or less than 0 since the equation has at most one root. So we have $4 \cdot \left(\int_E \sqrt{f(x)g(x)}dx \right)^2 - 4 \int_E f(x)dx \cdot \int_E g(x)dx \leqq 0$. Thereore,

$$\left(\int_E \sqrt{f(x)g(x)}dx \right)^2 \leqq \int_E f(x)dx \cdot \int_E g(x)dx$$

STEP 3. If $f(x)g(x) \geqq 1$, $\sqrt{f(x)g(x)} \geqq 1$. So we have

$$1 = (m(E))^2 \leqq \left(\int_E \sqrt{f(x)g(x)}dx \right)^2 \leqq \int_E f(x)dx \cdot \int_E g(x)dx.$$

□

117 (Exercise 7) Since $g(x) \in L(\mathbb{R}^d)$, we have $\int_E ((f(x) - g(x)) + g(x))\, dx = \int_E (f(x) - g(x))dx + \int_E g(x)dx$. (Let us recall that $\int_E (f_1 + f_2) = \int_E f_1 + \int_E f_2$ if at least one of f_1, f_2 is integrable on E.) $0 \leqq f(x) - g(x) \leqq h(x) - g(x)$ and $0 \leqq \int_E (h(x) - g(x))dx < \epsilon$, so $f(x) - g(x)$ is integrable. Therefore $f(x) = (f(x) - g(x)) + g(x)$ is integrable. □

118 (Exercise 8) We can take a subsequence of E_k s.t $\int_{\mathbb{R}^d} \left| \chi_{E_{k(m)}}(x) - f(x) \right| dx < \frac{1}{m^2}$. So $\sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \left| \chi_{E_{k(m)}}(x) - f(x) \right| dx < \infty$. Since $\left| \chi_{E_{k(m)}}(x) - f(x) \right|$ is non-negative, we may swap $\sum_{m=1}^{\infty}$ and $\int_E$. And we have

$$\int_{\mathbb{R}^d} \sum_{m=1}^{\infty} \left| \chi_{E_{k(m)}}(x) - f(x) \right| dx < \infty.$$

This implies that $\sum_{m=1}^{\infty} \left| \chi_{E_{k(m)}}(x) - f(x) \right| < \infty$ a.e $x \in E$. (When $f(x)$ is integrable on E, $|f(x)| < \infty$ a.e $x \in E$.) Since the infinite series converges, $\lim_{m\to\infty} \left| \chi_{E_{k(m)}}(x) - f(x) \right| = 0$ a.e $x \in E$. So $\limsup_{m\to\infty} \chi_{E_{k(m)}}(x) = f(x)$ a.e $x \in E$. (Since the limit exists a.e $x \in E$, so $\liminf(\cdots) = \limsup(\cdots)$ a.e $x \in E$. Therefore we may change it to $\limsup(\cdots)$ or $\liminf(\cdots)$. Here we change it to $\limsup(\cdots)$.) We have already discussed $\limsup_{k\to\infty} \chi_{E_k}(x) = \chi_{\limsup_{k\to\infty} E_k}(x)$. So

$$E = \limsup_{m\to\infty} E_{k(m)} \in \mathscr{M}$$

is the desired measurable set. □

119 (Exercise 9) Let $E_1 \overset{\text{def}}{=} [0,t] \cap E$, $E_2 \overset{\text{def}}{=} [0,t] \setminus E$ and $E_3 \overset{\text{def}}{=} (t,1] \cap E$. Then $[0,t] = E_1 \cup E_2$ and $E = E_1 \cup E_3$. Since $m(E) = t$, we have $m(E_2) = m(E_3)$. ($f(x)$ is bounded. So the following integrals are all finite.)

$$\begin{aligned}\int_{[0,t]} f(x)dx = \int_{E_1\cup E_2} f(x)dx &= \int_{E_1} f(x)dx + \int_{E_2} f(x)dx \\ &\overset{*1}{\leqq} \int_{E_1} f(x)dx + \int_{E_2} f(t)dx \\ &= \int_{E_1} f(x)dx + f(t)\cdot m(E_2) \\ &\overset{*2}{=} \int_{E_1} f(x)dx + f(t)\cdot m(E_3) \\ &= \int_{E_1} f(x)dx + \int_{E_3} f(t)dx \\ &\overset{*3}{\leqq} \int_{E_1} f(x)dx + \int_{E_3} f(x)dx \\ &= \int_{E_1\cup E_3} f(x)dx \\ &= \int_{E} f(x)dx\end{aligned}$$

- $(*1)$ $f(x) \leqq f(t)$ on $x \in E_2 \subset [0,t]$.
- $(*2)$ $m(E_2) = m(E_3)$
- $(*3)$ $f(t) \leqq f(x)$ on $x \in E_3 \subset (t,1]$

□

120 (Exercise 10)

STEP 1. Since $|f(x)|\chi_{x\in\mathbb{R}^d||x|>r}(x) \leqq |f(x)| \in L(\mathbb{R}^d)$, we can apply Lebesgue Dominated Convergence Theorem.

$$\begin{aligned}\lim_{r\to\infty}\int_{\{x\in\mathbb{R}^d||x|>r\}} |f(x)|\,dx &= \lim_{r\to\infty}\int_{\mathbb{R}^d} |f(x)|\cdot\chi_{\{x\in\mathbb{R}^d||x|>r\}}(x)dx \\ &= \int_{\mathbb{R}^d}\lim_{r\to\infty} |f(x)|\cdot\chi_{\{x\in\mathbb{R}^d||x|>r\}}(x)dx \\ &\overset{*1}{=} \int_{\mathbb{R}^d} 0\ dx = 0\end{aligned}$$

- $(*1)$ Suppose that $x_0 \in \mathbb{R}^d$. When r is sufficiently large, $r > |x_0|$. so $\chi_{\{x\in\mathbb{R}^d||x|>r\}}(x_0) = 0$.

STEP 2. Since E is bounded so we suppose that $E \subset B(0,M)$. Let $x \in E_{+y}$. Then there exists $z \in E$ s.t $x = y+z$. By triangular inequality, $|x| = |y+z| \geqq |y|-|z| \geqq |y|-M$.

This implies that $x \in E_{+y} \subset \{x \in \mathbb{R}^d \mid |x| \geqq |y| - M\}$.

$$\begin{aligned} \limsup_{|y|\to\infty} \int_{E_{+y}} |f(x)|dx &\leqq \lim_{|y|\to\infty} \int_{\{x\in\mathbb{R}^d \mid |x|\geqq|y|-M\}} |f(x)|dx \\ &= \lim_{r\to\infty} \int_{\{x\in\mathbb{R}^d \mid |x|>r\}} |f(x)|\, dx = 0 \end{aligned}$$

□

121 (Exercise 11)

(1)

STEP 1. $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ if $|r| < 1$. When $x \in (0, \infty)$, $0 < \exp(-x) < 1$. So we have

$$\begin{aligned} \frac{x^{\alpha-1}}{\exp(x) - 1} &= \frac{x^{\alpha-1}\exp(-x)}{1 - \exp(-x)} \\ &= x^{\alpha-1}\exp(-x) \cdot \frac{1}{1-\exp(-x)} \\ &= x^{\alpha-1}\exp(-x) \cdot \sum_{n=0}^{\infty} \exp(-nx) \\ &= \sum_{n=1}^{\infty} x^{\alpha-1} \cdot \exp(-nx) \end{aligned}$$

.

STEP 2. Since $x^{\alpha-1} \cdot \exp(-nx)$ is non-negative for all $n \geqq 1$, by Theorem 4.6 we have,

$$\begin{aligned} \int_{(0,\infty)} \frac{x^{\alpha-1}}{\exp(x)-1} dx &= \int_{(0,\infty)} \sum_{n=1}^{\infty} x^{\alpha-1} \cdot \exp(-nx) dx \\ &= \sum_{n=1}^{\infty} \int_{(0,\infty)} x^{\alpha-1} \cdot \exp(-nx) dx. \end{aligned}$$

By monotone convergence theorem and §4.2 Example 10, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\int_{(0,\infty)} x^{\alpha-1}\cdot\exp(-nx)dx &= \sum_{n=1}^{\infty}\lim_{c\to\infty}\int_{(0,c)} x^{\alpha-1}\cdot\exp(-nx)dx \\
&= \sum_{n=1}^{\infty}\lim_{c\to\infty}\int_{(0,c)} \left(\frac{t}{n}\right)^{\alpha-1}\cdot\frac{1}{n}\cdot\exp(-t)dt \\
&= \sum_{n=1}^{\infty}\int_{(0,\infty)} \left(\frac{t}{n}\right)^{\alpha-1}\cdot\frac{1}{n}\cdot\exp(-t)dt \\
&= \sum_{n=1}^{\infty}\int_{(0,\infty)} \left(\frac{1}{n}\right)^{\alpha}\cdot t^{\alpha-1}\cdot\exp(-t)dt \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{\alpha}\cdot\Gamma(\alpha)
\end{aligned}$$

In the equations above, we used the fact that

$$\begin{aligned}
\Gamma(\alpha) \overset{\text{def}}{=} (\text{R})\int_{[0,\infty)} t^{\alpha-1}\exp(-t)dt &= (\text{R})\ \lim_{c\to\infty}\int_{[0,c]} t^{\alpha-1}\exp(-t)dt \\
&\overset{*}{=} (\text{L})\ \lim_{c\to\infty}\int_{[0,c]} t^{\alpha-1}\exp(-t)dt \\
&= (\text{L})\int_{[0,\infty)} t^{\alpha-1}\exp(-t)dt \\
&= (\text{L})\int_{(0,\infty)} t^{\alpha-1}\exp(-t)dt
\end{aligned}$$

- ($*$) On $[0,c]$, $t^{\alpha-1}\exp(-t)$ is continuous so it is Riemann integrable ($\because$ continuous a.e $[0,c]$) and its integral is same as Lebesgue integral.

(2)

STEP 1.

$$\begin{aligned}
\frac{\sin ax}{\exp(x)-1} &= \sin ax\cdot\frac{\exp(-x)}{1-\exp(-x)} \\
&= \sin ax\cdot\exp(-x)\sum_{n=0}^{\infty}\exp(-nx) \\
&= \sum_{n=1}^{\infty}\sin ax\cdot\exp(-nx)
\end{aligned}$$

STEP 2.

$$\begin{aligned}
\left|\sum_{n=1}^{k}\sin ax\cdot\exp(-nx)\right| &\leqq \sum_{n=1}^{k}|\sin ax|\cdot\exp(-nx) \\
&\leqq \sum_{n=1}^{\infty}x\cdot\exp(-nx)\overset{*}{\in} L(0,\infty).
\end{aligned}$$

- (*) $\int_{(0,\infty)} \sum_{n=1}^{\infty} x \cdot \exp(-nx)dx = \sum_{n=1}^{\infty} \int_{(0,\infty)} x \cdot \exp(-nx)dx = \sum_{n=1}^{\infty} \int_{(0,\infty)} \frac{t}{n^2} \cdot \exp(-t)dt = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$.

STEP 3. Let us apply Lebesgue Dominated Convergence Theorem to $g_k(x) \overset{\text{def}}{=} \sum_{n=1}^{k} \sin ax \cdot \exp(-nx)$.

$$\begin{aligned}
\int_{(0,\infty)} \frac{\sin ax}{\exp(x)-1} dx &= \int_{(0,\infty)} \lim_{k\to\infty} g_k(x)dx \\
&= \lim_{k\to\infty} \int_{(0,\infty)} g_k(x)dx \\
&= \lim_{k\to\infty} \int_{(0,\infty)} \sum_{n=1}^{k} \sin ax \cdot \exp(-nx)dx \\
&= \lim_{k\to\infty} \sum_{n=1}^{k} \int_{(0,\infty)} \sin ax \cdot \exp(-nx)dx \\
&= \sum_{n=1}^{\infty} \int_{(0,\infty)} \sin ax \cdot \exp(-nx)dx \\
&= \sum_{n=1}^{\infty} \int_{[0,\infty)} \sin ax \cdot \exp(-nx)dx
\end{aligned}$$

STEP 4. We find $\int_{[0,\infty)} \sin ax \cdot \exp(-nx)dx$. Since $|\sin ax \cdot \exp(-nx)| \leqq \exp(-x) \in L([0,\infty)) \subset L([0,c]),\ 0 < c < \infty$, by Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}
\lim_{c\to\infty} \int_{[0,c]} \sin ax \cdot \exp(-nx)dx &= \lim_{c\to\infty} \int_{[0,\infty)} \sin ax \cdot \exp(-nx) \cdot \chi_{[0,c]}(x)dx \\
&= \int_{[0,\infty)} \lim_{c\to\infty} \sin ax \cdot \exp(-nx) \cdot \chi_{[0,c]}(x)dx \\
&= \int_{[0,\infty)} \sin ax \cdot \exp(-nx)dx
\end{aligned}$$

Since Riemann integrable implies Lebesgue integrable, we have

$$\begin{aligned}
\text{(R)} \int_{[0,c]} \sin ax \cdot \exp(-nx)dx &= \frac{a}{n^2+a^2} - \frac{1}{n^2+a^2}(n \sin ac + a \cos ac) \cdot \exp(-nc) \\
&= \text{(L)} \int_{[0,c]} \sin ax \cdot \exp(-nx)dx.
\end{aligned}$$

So

$$\begin{aligned}
&\lim_{c\to\infty} \int_{[0,c]} \sin ax \cdot \exp(-nx)dx \\
= &\lim_{c\to\infty} \left(\frac{a}{n^2+a^2} - \frac{1}{n^2+a^2}(n \sin ac + a \cos ac) \cdot \exp(-nc) \right) \\
= &\frac{a}{n^2+a^2}.
\end{aligned}$$

Now the proof is complete.

□

122 (Exercise 12) Let $S(x) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} f\left(\frac{x}{a}+n\right)$. First we show that $S(x)$ converges absolutely a.e $x \in [0,a]$. Let us consider

$$\int_{[0,a]} \sum_{n=-\infty}^{\infty} \left| f\left(\frac{x}{a}+n\right)\right| dx.$$

Since $|f\left(\frac{x}{a}+n\right)|$ is non-negative, we may swap $\int$ and $\sum$ by Theoerm 4.6, we have

$$\sum_{n=-\infty}^{\infty} \int_{[0,a]} \left| f\left(\frac{x}{a}+n\right)\right| dx.$$

By §4.2 Example 10,

$$\sum_{n=-\infty}^{\infty} \int_{[0,a]} \left| f\left(\frac{x}{a}+n\right)\right| dx = \sum_{n=-\infty}^{\infty} \int_{[0,1]} a \cdot |f(y+n)|dy.$$

Furthermore,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \int_{[0,1]} a \cdot |f(y+n)| &= \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} a \cdot |f(y+n)|\chi_{[0,1]}(y)dy \\
&= \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} a \cdot |f(y)|\chi_{[0,1]}(y-n)dy \\
&= \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} a \cdot |f(y)|\chi_{[n,n+1]}(y)dy \\
&= \sum_{n=-\infty}^{\infty} \int_{[n,n+1]} a \cdot |f(y)|dy \\
&= \sum_{n=-\infty}^{\infty} \int_{[n,n+1)} a \cdot |f(y)|dy \\
&= \int_{(-\infty,\infty)} a \cdot |f(y)|dy < \infty
\end{aligned}$$

because $f(x) \in L(\mathbb{R})$. Therefore we conclude that $\sum_{n=-\infty}^{\infty} |f\left(\frac{x}{a}+n\right)| \in L([0,a])$ and thus $\sum_{n=-\infty}^{\infty} |f\left(\frac{x}{a}+n\right)| < \infty$ a.e $x \in [0,a]$. So $S(x)$ converges absolutely a.e $x \in [0,a]$. It is easy to find out that $S(x) = S(x+ka)$, $k \in \mathbb{Z}$, so $S(x)$ converges absolutely a.e $x \in \mathbb{R}$. □

123 (Exercise 13) Let us consider $\int_{\mathbb{R}} \sum_{n=1}^{\infty} n^{-p} |f(nx)| \, dx$.

$$\begin{aligned}
\int_{\mathbb{R}} \sum_{n=1}^{\infty} n^{-p} |f(nx)| \, dx &\overset{*1}{=} \sum_{n=1}^{\infty} \int_{\mathbb{R}} n^{-p} |f(nx)| \, dx \\
&\overset{*2}{=} \sum_{n=1}^{\infty} \lim_{c\to\infty} \int_{(-c,c)} n^{-p} |f(nx)| \, dx \\
&\overset{*3}{=} \sum_{n=1}^{\infty} \lim_{c\to\infty} \int_{(-nc,nc)} n^{-p} \frac{1}{n} |f(y)| \, dy \\
&\overset{*4}{=} \sum_{n=1}^{\infty} \int_{(-\infty,\infty)} n^{-p} \frac{1}{n} |f(y)| \, dy \\
&\overset{*5}{=} \sum_{n=1}^{\infty} \int_{(-\infty,\infty)} \frac{1}{n^{1+p}} |f(y)| \, dy \\
&\overset{*6}{=} \sum_{n=1}^{\infty} \frac{1}{n^{1+p}} \int_{(-\infty,\infty)} |f(y)| \, dy \\
&\overset{*7}{<} \infty
\end{aligned}$$

- (∗1) Theorem 4.6.
- (∗2) monotone convergence theorem to $n^{-p} |f(nx)| \cdot \chi_{(-c,c)}(x)$.
- (∗3) §4.2 Example 10
- (∗4) monotone convergence theorem.
- (∗5) obvious.
- (∗6) obvious. (linearity of integral)
- (∗7) $f \in L(\mathbb{R})$, $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} < \infty$ when $\alpha > 1$.

□

124 (Exercise 14) Let

$$g(u) \overset{\text{def}}{=} \int_{[0,\infty)} x^u f(x) dx.$$

If $x > 0$, then $x^u|f(x)| \leqq x^s|f(x)| + x^t|f(x)| \in L([0,\infty))$. (It is easy to verify this fact by considering $0 < x < 1$ and $x \geqq 1$) □

So $x^u f(x)$ is integrable hence $g(u)$ is well-defined. Next we prove that $g(u)$ is continuous. Consider $\{u_k\}_{k\geqq 1} \subset (s,t)$ s.t $u_k \to u \in (s,t)$. By the previous inequality

$x^{u_k}|f(x)| \leqq x^s|f(x)| + x^t|f(x)| \in L([0,\infty))$. By Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}\lim_{k\to\infty} g(u_k) &= \lim_{k\to\infty}\int_{[0,\infty)} x^{u_k}f(x)dx \\ &= \int_{[0,\infty)} \lim_{k\to\infty} x^{u_k}f(x)dx \\ &= \int_{[0,\infty)} x^u f(x)dx = g(u)\end{aligned}$$

Now the proof is complete.

125 (Exercise 15)

STEP 1. Let $k \in \mathbb{N}$.

$$\begin{aligned} c &= \int_{[0,1]} (f(x))^n dx \\ &= \int_{\{x\in[0,1]|f(x)>1+\frac{1}{k}\}} (f(x))^n dx + \int_{\{x\in[0,1]|0<f(x)\leqq 1+\frac{1}{k}\}} (f(x))^n dx \\ &\geqq \int_{\{x\in[0,1]|f(x)>1+\frac{1}{k}\}} \left(1+\frac{1}{k}\right)^n dx \\ &= m\left(\left\{x\in[0,1] \mid f(x) > 1+\frac{1}{k}\right\}\right)\cdot\left(1+\frac{1}{k}\right)^n dx, \forall n\in N\end{aligned}$$

If $m\left(\left\{x\in[0,1] \mid f(x) > 1+\frac{1}{k}\right\}\right) > 0$, the right hand side goes to infinity by taking $n\to\infty$. So $m\left(\left\{x\in[0,1] \mid f(x) > 1+\frac{1}{k}\right\}\right) = 0$ for all $k\in\mathbb{N}$. Moreover,

$$\begin{aligned} & m\left(\bigcup_{k=1}^{\infty}\left\{x\in[0,1] \mid f(x) > 1+\frac{1}{k}\right\}\right) \\ = \ & m\left(\{x\in[0,1] \mid f(x) > 1\}\right) = 0.\end{aligned}$$

This implies that $0 < f(x) \leqq 1$ a.e $x\in[0,1]$.

STEP 2. Since $0 < f(x) \leqq 1$ a.e $x\in[0,1] \Rightarrow 0 < (f(x))^n \leqq 1$ a.e $x\in[0,1]$ and $1\in L([0,1])$, by Lebesgue Dominated Convergence theorem, we have

$$\begin{aligned} c = \lim_{n\to\infty}\int_{[0,1]} (f(x))^n dx &= \int_{[0,1]} \lim_{n\to\infty} (f(x))^n dx \\ &= \int_{[0,1]} \chi_{\{x\in[0,1]|f(x)=1\}}(x)dx \\ &= m(\{x\in[0,1] \mid f(x) = 1\})\end{aligned}$$

STEP 3.

$$\begin{aligned} c &= \int_{[0,1]} (f(x))^n dx \\ &= \int_{\{x\in[0,1]|0<f(x)<1\}} (f(x))^n dx + \int_{\{x\in[0,1]|f(x)=1\}} (f(x))^n dx \\ &= \int_{\{x\in[0,1]|0<f(x)<1\}} (f(x))^n dx + m(\{x \in [0,1] \mid f(x) = 1\}) \\ &= \int_{\{x\in[0,1]|0<f(x)<1\}} (f(x))^n dx + c \end{aligned}$$

So we have $\int_{\{x\in[0,1]|0<f(x)<1\}}(f(x))^n dx = 0$. Since $f(x) > 0$, we have $m(\{x \in [0,1] \mid 0 < f(x) < 1\}) = 0$. This implies that $f(x) = 1$ a.e $x \in [0,1]$ hence $c = 1$.

□

126 (Exercise 16)

STEP 1. Let $x \in [0,1]$. Then $\exp(x) - x - 1 \geqq 0$. So $\exp(x) \geqq x + 1$. By taking $\ln(\cdot)$ of the both sides, we have $x \geqq \ln(x+1)$. Since $x \geqq x^2$, we have $\ln(x+1) \geqq \ln(x^2+1)$. So $x \geqq \ln(x^2 + 1)$.

STEP 2. When n is sufficiently large, $\frac{|f(x)|}{n} \in [0,1]$ a.e $x \in [0,1]$ because $|f(x)| \in L([0,1])$ implies that $|f(x)| < \infty$ a.e $x \in [0,1]$. By the inequality above, when n is sufficiently large, we have $\frac{|f(x)|}{n} \geqq \ln\left(\frac{|f(x)|^2}{n^2} + 1\right)$ a.e $x \in [0,1]$. By multiplying n to the both sides, we have $|f(x)| \geqq n \cdot \ln\left(\frac{|f(x)|^2}{n^2} + 1\right)$ and the left side is integrable on $[0,1]$ hence we may apply Lebesgue Dominated Convergence Theorem.

STEP 3. By Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \int_{[0,1]} n \cdot \ln\left(\frac{|f(x)|^2}{n^2} + 1\right) dx &\leqq \int_{[0,1]} \lim_{n\to\infty} n \cdot \ln\left(\frac{|f(x)|^2}{n^2} + 1\right) dx \\ &= \int_{[0,1]} \lim_{n\to\infty} \frac{|f(x)|^2}{n} \cdot \frac{n^2}{|f(x)|^2} \cdot \ln\left(\frac{|f(x)|^2}{n^2} + 1\right) dx \\ &= \int_{[0,1]} \lim_{n\to\infty} \frac{|f(x)|^2}{n} \cdot \ln\left(\frac{|f(x)|^2}{n^2} + 1\right)^{\frac{n^2}{|f(x)|^2}} dx \\ &= \int_{[0,1]} 0 \cdot \ln(e)\, dx = \int_{[0,1]} 0\, dx = 0. \end{aligned}$$

□

127 (Exercise 17) $|f(x)| \cdot \chi_{E_k}(x) \leqq |f(x)| \cdot \chi_{E_1}(x) \in L(E_1)$. By Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}\lim_{k\to\infty}\int_{E_k} f(x)dx &= \lim_{k\to\infty}\int_{E_1} f(x)\cdot\chi_{E_k}(x)dx\\ &= \int_{E_1}\lim_{k\to\infty} f(x)\cdot\chi_{E_k}(x)dx\\ &= \int_{E_1} f(x)\cdot\chi_E(x)dx\\ &= \int_E f(x)dx\end{aligned}$$

□

128 (Exercise 18)

STEP 1. $(m(E)=\infty)$ By Fatou's lemma, we have

$$\int_E \liminf_{k\to\infty}(f(x))^{1/k}dx \leqq \liminf_{k\to\infty}\int_E (f(x))^{1/k}dx.$$

Since $f(x)>0$, the left hand side is $\int_E 1\ dx = m(E)=\infty$. So we have

$$\lim_{k\to\infty}\int_E (f(x))^{1/k}dx=\infty.$$

STEP 2. $(m(E)<\infty)$ We separate $\{x\in E\mid f(x)>1\}$ and $\{x\in E\mid 0<f(x)\leqq 1\}$. Let $p\in(0,1]$. If $a>1$, then $a^p\leqq a$ and if $0<a\leqq 1$, then $a\leqq a^p\leqq 1$. So we have

$$\begin{aligned}(f(x))^{1/k} &= (f(x))^{1/k}\cdot\chi_{\{x\in E|f(x)>1\}}(x)+(f(x))^{1/k}\cdot\chi_{\{x\in E|0<f(x)\leqq 1\}}(x)\\ &\leqq f(x)\cdot\chi_{\{x\in E|f(x)>1\}}(x)+1\cdot\chi_{\{x\in E|0<f(x)\leqq 1\}}(x)\\ &\leqq f(x)+1\in L(E)\end{aligned}$$

By Lebesgue Dominated Convergence Theorem, we have the desired conclusion.

□

129 (Exercise 19) The proof is not easy. This exercies is related to L^p convergence, absolute continuity, and uniform integrability. □

130 (Exercise 20) Let $g_k(x)\overset{\text{def}}{=}\max\{f_1(x),f_2(x),\cdots,f_k(x)\}$. Then $g_k(x)\leqq g_{k+1}(x)$ and $g_k(x)$ is non-negative. We apply monotone convergence theorem to $g_k(x)$. We have

$$\begin{aligned}\lim_{k\to\infty}\int_E g_k(x)dx &= \int_E\lim_{k\to\infty} g_k(x)dx\\ &= \int_E\sup_{k\geqq 1}\{f_k(x)\}dx\\ &\overset{*}{\leqq} M<\infty\end{aligned}$$

Therefore $\sup_{k\geqq 1}\{f_k(x)\}$ is integrable.

- $(*)$ holds because $\int_E g_k(x)dx \leqq M$ so $\sup_{k\geqq 1}\int_E g_k(x)dx \leqq M$.

Since $0 \leqq f_k(x) \leqq \sup_{k\geqq 1}\{f_k(x)\} \in L(E)$, we can apply Lebesgue Dominated Convergence Theorem. And we have the desired conclusion. □

131 (Exercise 21)

STEP 1. Let $\{a_n\}_{n\geqq 1}$ be a sequence of real numbers. We show that we can find a subsequence n_k s.t

$$\lim_{k\to\infty} a_{n_k} = \liminf_{n\to\infty} a_n.$$

case 1. ($\liminf_{n\to\infty} a_n = \infty$) This implies that $a_n \to \infty$ so we can let $n_k = k$.

case 2. ($\liminf_{n\to\infty} a_n = -\infty$) $\lim_{k\to\infty}\inf_{n\geqq k} a_n = -\infty$. Since $\inf_{n\geqq k} a_n = -\infty$ for all k, we can find n_k s.t $a_{n_k} < -k$.

case 3. ($\liminf_{n\to\infty} a_n \in (-\infty,\infty)$) Let $a \overset{\text{def}}{=} \liminf_{n\to\infty} a_n$. $b_k \overset{\text{def}}{=} \inf_{n\geqq k} a_n$ is an increasing sequence with respect to k and $b_k \nearrow a$. We can find a subsequence k_ℓ s.t $0 \leqq a - b_{k_\ell} < \frac{1}{2\ell}$. Since $b_{k_\ell} = \inf_{n\geqq k_\ell} a_n$, we can find $k^*_\ell \geqq k_\ell$ s.t $0 \leqq a_{k^*_\ell} - b_{k_\ell} < \frac{1}{2\ell}$. So $|a - a_{k^*_\ell}| \leqq |a - b_{k_\ell}| + |a_{k^*_\ell} - b_{k_\ell}| < \frac{1}{\ell}$. So $a_{k^*_\ell} \to a$.

STEP 2. We can find a subsequence $\{k_\ell\}_{\ell\geqq 1}$ s.t

$$\lim_{\ell\to\infty}\int_E f_{k_\ell}(x)dx = \liminf_{k\to\infty}\int_E f_k(x)dx \ (*)$$

Let us recall that $f_k(x) \xrightarrow{m} f(x)$ if and only if $\forall k_\ell\ \exists k_{\ell_m}$ s.t $f_{k_{\ell_m}}(x) \xrightarrow{\text{a.u}} f(x)$ and that $\xrightarrow{\text{a.u}}$ implies $\xrightarrow{\text{a.e}}$. So we can find a subsubsequence k_{ℓ_m} s.t $f_{k_{\ell_m}}(x) \xrightarrow{\text{a.e}} f(x)$. We apply Fatou's lemma to $f_{k_{\ell_m}}(x)$ and we obtain

$$\begin{aligned}\int_E \liminf_{m\to\infty} f_{k_{\ell_m}}(x)dx &\leqq \liminf_{m\to\infty}\int_E f_{k_{\ell_m}}(x)dx \\ &\overset{*1}{=} \lim_{\ell\to\infty}\int_E f_{k_\ell}(x)dx \\ &\overset{*2}{=} \liminf_{k\to\infty}\int_E f_k(x)dx\end{aligned}$$

- $(*1)$ $m \to \infty \Rightarrow \ell_m \to \infty$.
- $(*2)$ See $(*)$.

The left hand side is $\int_E f(x)dx$. So we have the desired conclusion.

□

132 (Exercise 22) Let us apply Theorem 4.17.

STEP 1. Let

$$f(x,t) \overset{\text{def}}{=} \exp(-x^2)\cos 2tx,\ (x,t) \in [0,\infty)\times(-\infty,\infty).$$

The partial derivative of $f(x,t)$ with respect to t is

$$\frac{\partial}{\partial t}f(x,t) = -2x\exp(-x^2)\sin 2tx$$

And

$$\left|\frac{\partial}{\partial t}f(x,t)\right| \leqq 2x\exp(-x^2) \overset{(*1)}{\in} L([0,\infty)).$$

We explain why $(*1)$ holds. We know that the improper Riemann integral

$$\text{(R)} \int_0^\infty 2x\exp(-x^2)dx = \text{(R)} \lim_{c\to\infty}\int_0^c 2x\exp(-x^2)dx < \infty.$$

For each $0 < c < \infty$,

$$\text{(L)} \int_{[0,c]} 2x\exp(-x^2)dx = \text{(R)} \int_0^c 2x\exp(-x^2)dx$$

because the right hand side is Riemann integrable. By taking $c \to \infty$ and applying monotone convergence theorem, we have

$$\text{(L)} \int_{[0,\infty)} 2x\exp(-x^2)dx = \text{(R)} \int_0^\infty 2x\exp(-x^2)dx < \infty.$$

STEP 2. Let

$$g(t) \overset{\text{def}}{=} \int_{[0,\infty)} f(x,t)dx.$$

And

$$\begin{aligned}
g'(t) &\overset{*2}{=} \int_{[0,\infty)} \frac{\partial}{\partial t}f(x,t)dx \\
&= \int_{[0,\infty)} -2x\exp(-x^2)\sin 2txdx \\
&\overset{*3}{=} \lim_{c\to\infty}\int_{[0,\infty)} -2x\exp(-x^2)\sin 2tx \cdot \chi_{[0,c]}(x)dx \\
&= \lim_{c\to\infty}\int_{[0,c]} -2x\exp(-x^2)\sin 2txdx
\end{aligned}$$

- $(*2)$ By Theorem 4.17, we may swap $\frac{\partial}{\partial t}$ and $\int$.
- $(*3)$ Lebesgue Dominated Convergence Theorem. $\left|-2x\exp(-x^2)\cdot\chi_{[0,c]}(x)\right| \leqq 2x\exp(-x^2) \in L([0,\infty))$.

We find the above integral

$$\text{(L)} \int_{[0,c]} -2x\exp(-x^2)\sin 2xtdx$$

using Riemann integral. (We have already learned it in basic caluclus.) $\int_{[0,c]} -2x\exp(-x^2)\sin 2xtdx$ is Riemann integrable because this is a continuous function on $[0,c]$.

$$(\mathrm{R})\int_0^c -2x\exp(-x^2)\sin 2xtdx = (\mathrm{R})\ \exp(-c^2) - 2t\int_0^c \exp(-x^2)\cos 2xtdx$$

And the Riemann integrals (R) $\int_0^c -2x\exp(-x^2)\sin 2xtdx$ and (R) $\int_0^c \exp(-x^2)\cos 2xtdx$ are equal to Lebesgue integrals. Therefore we have

$$\begin{aligned} g'(t) &= \lim_{c\to\infty}\int_{[0,c]} -2x\exp(-x^2)\sin 2txdx \\ &= \lim_{c\to\infty}\left(\exp(-c^2) - 2t\int_{[0,c]}\exp(-x^2)\cos 2xtdx\right) \\ &= \lim_{c\to\infty} -2t\cdot\int_{[0,c]}\exp(-x^2)\cos 2xtdx \\ &\overset{*4}{=} -2t\cdot\int_{[0,\infty)}\exp(-x^2)\cos 2xtdx \\ &= -2t\cdot g(t). \end{aligned}$$

- (∗4) Lebesgue Dominated Convergence Theorem. $|\exp(-x^2)\cos 2xt|\cdot\chi_{[0,c]}(x) \leqq \exp(-x^2) \in L([0,\infty))$.

By solving the differential equation, we have $g(t) = g(0)\cdot\exp(-t^2)$. And $g(0) = \frac{\sqrt{\pi}}{2}$. Now the proof is complete.

□

133 (Exercise 23)

STEP 1. As with Exercise 5, $f_1(x) \leqq f_2(x) \leqq \cdots \leqq f_k(x) \leqq \cdots$ a.e $x \in \mathbb{R}^d$. From this fact, $f_k(x)$ converges a.e $x \in \mathbb{R}^d$ because it is monotone increasing a.e $x \in \mathbb{R}^d$. Therefore $\tilde{f}(x)$ is measurable.

$$\tilde{f}(x) \overset{\text{def}}{=} \begin{cases} \lim_{k\to\infty} f_k(x) & \text{if } f_k(x) \text{ converges} \\ 0 & \text{otherwise} \end{cases}$$

STEP 2. Let $g_k(x) \overset{\text{def}}{=} f_k(x) - f_1(x) \geqq 0$ a.e $x \in \mathbb{R}^d$. $g_k(x) \overset{\text{a.e}}{\longrightarrow} \tilde{f}(x) - f_1(x)$ a.e $x \in \mathbb{R}^d$ (hence $x \in E \in \mathscr{M}$). We apply monotone convergence theorem to $g_k(x)$.

$$\lim_{k\to\infty}\int_E g_k(x)dx = \int_E \lim_{k\to\infty} g_k(x)dx,$$

where $E \in \mathscr{M}$ is an arbitrary Lebesgue measurable set on $\mathbb{R}^d$. The left hand side is

$$\begin{aligned}
\lim_{k\to\infty}\int_E g_k(x)dx &= \lim_{k\to\infty}\int_E (f_k(x)-f_1(x))dx \\
&\overset{*1}{=} \lim_{k\to\infty}\left(\int_E f_k(x)dx - \int_E f_1(x)dx\right) \\
&= \lim_{k\to\infty}\int_E f_k(x)dx - \int_E f_1(x)dx \\
&\overset{*2}{=} \int_E f(x)dx - \int_E f_1(x)dx
\end{aligned}$$

- ($*1$) $f_1(x), f_k(x) \in L(\mathbb{R}^d)$ so linearity holds. (When at least one of f_1, f_2 is integrable, $\int_E(f_1+f_2) = \int_E f_1 + \int_E f_2$.)
- ($*2$) By assumption.

The right hand side is

$$\begin{aligned}
\int_E \lim_{k\to\infty} g_k(x)dx &= \int_E (\tilde{f}(x) - f_1(x))dx \\
&\overset{(*2)}{=} \int_E \tilde{f}(x)dx - \int_E f_1(x)dx
\end{aligned}$$

- ($*2$) $f_1(x) \in L(\mathbb{R}^d)$ so linearity holds.

By adding $\int_E f_1(x)dx$ to the both sides, we have

$$\int_E \tilde{f}(x)dx = \int_E f(x)dx$$

for all $E \in \mathscr{M}, E \subset \mathbb{R}^d$.

STEP 3. The integrals above are finite by assumption, so we can subtract one from another. And the integral has linearity, so we have

$$\int_E (\tilde{f}(x) - f(x))dx = 0.$$

for all $E \in \mathscr{M}$. Let $E = \{x \in \mathbb{R}^d \mid \tilde{f}(x) - f(x) > 0\}$. And we have $m(\{x \in \mathbb{R}^d \mid \tilde{f}(x) - f(x) > 0\}) = 0$. Similarly, we have $m(\{x \in \mathbb{R}^d \mid \tilde{f}(x) - f(x) < 0\}) = 0$. So $\tilde{f}(x) = f(x)$ a.e $x \in \mathbb{R}^d$.

□

134 (Exercise 24)

STEP 1. Let $\{a_n\}, \{b_n\}$ be sequences of real numbers. Suppose that $a_n \to a \in (-\infty, \infty)$. Then $\liminf_{n\to\infty}(a_n + b_n) = a + \liminf_{n\to\infty} b_n$. First we prove this fact. Since

$$\lim_{n\to\infty}(\inf_{m'\geqq n} a_{m'} + \inf_{m\geqq n} b_m) \leqq \lim_{n\to\infty}\inf_{m\geqq n}(a_m + b_m) \leqq \lim_{n\to\infty}(\sup_{m'\geqq n} a_{m'} + \inf_{m\geqq n} b_m),$$

we have

$$\lim_{n\to\infty}\inf_{m'\geqq n} a_{m'} + \lim_{n\to\infty}\inf_{m\geqq n} b_m \leqq \lim_{n\to\infty}\inf_{m\geqq n}(a_m+b_m) \leqq \lim_{n\to\infty}\sup_{m'\geqq n} a_{m'} + \lim_{n\to\infty}\inf_{m\geqq n} b_m.$$

Since $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a$,

$$a + \lim_{n\to\infty}\inf_{m\geqq n} b_m \leqq \lim_{n\to\infty}\inf_{m\geqq n}(a_m+b_m) \leqq a + \lim_{n\to\infty}\inf_{m\geqq n} b_m.$$

Now we have the deisred conclusion.

STEP 2. We apply Fatou's lemma to $g_n(x) - f_n(x) \geqq 0$ and $g_n(x) + f_n(x) \geqq 0$.

$$\begin{aligned}
\int_E \liminf_{n\to\infty}(g_n(x) - f_n(x))dx &\leqq \liminf_{n\to\infty}\int_E (g_n(x) - f_n(x))dx \\
&\overset{*1}{=} \liminf_{n\to\infty}\left(\int_E g_n(x)dx - \int_E f_n(x)dx\right) \\
&\overset{*2}{=} \int_E g(x)dx + \liminf_{n\to\infty}\left(-\int_E f_n(x)dx\right) \\
&= \int_E g(x)dx - \limsup_{n\to\infty}\int_E f_n(x)dx
\end{aligned}$$

- (∗1) By assmption, $g_n(x)$ is integrable for sufficiently large n. So we may separate into two integrals.
- (∗2) Step 1.

By assumption, the left hand side is

$$\int_E (g(x) - f(x))dx.$$

And $g(x)$ is integrable, so the left hand side is

$$\int_E g(x)dx - \int_E f(x)dx.$$

By subtracting $\int_E g(x)dx$ (this is finite) from the both sides, we have

$$\limsup_{n\to\infty}\int_E f_n(x)dx \leqq \int_E f(x)dx.$$

Let us repeat the similar argument to $g_n(x) + f_n(x)$ and we have

$$\int_E f(x)dx \leqq \liminf_{n\to\infty}\int_E f_n(x)dx.$$

By merging these two results, we have the desired conclusion.

□

135 (Exercise 25) $D = D \setminus D' \cup D \cap D'$. Since $D \setminus D'$ is a set of isolated points, so $D \setminus D'$ is countable. And $D \cap D' \subset D'$, so this is also countable. Therefore D is countable. We conclude that $f(x)$ is Riemann integrable. □

136 (Exercise 26) It is enough for us to prove that

$$A \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid f \text{ is discontinuous at } x, \lim_{y \to x+0} f(y) \text{ exists}\}$$

is countable. □

137 (Exercise 27)

STEP 1. Let

$$\omega_f(x) \stackrel{\text{def}}{=} \lim_{\delta \searrow 0} \sup_{x', x'' \in B(x,\delta)} |f(x') - f(x'')|.$$

Let us recall that the points of discontinuity of f is

$$\{x \in [0,1] \mid \omega_f(x) > 0\}.$$

Let $f(x) \stackrel{\text{def}}{=} \chi_E(x)$. We prove that $\chi_{\overline{E} \setminus \mathring{E}}(x) = \omega_f(x)$. Then $\{x \in [0,1] \mid \omega_f(x) > 0\} = \{x \in [0,1] \mid \omega_f(x) = 1\} = \overline{E} \setminus \mathring{E}$.

STEP 2. Let $A \stackrel{\text{def}}{=} \omega_f(x)$, $B \stackrel{\text{def}}{=} \chi_{\overline{E} \setminus \mathring{E}}(x)$. We prove that $A \leqq B$ and $A \geqq B$. (We may suppose that $x \in [0,1]$ is fixed.)

First we show that $A \leqq B$. Since both A, B take only 0 or 1, it is enough to show that $A = 1 \Rightarrow B = 1$. Suppose that $A = 1$. Then $\forall \delta > 0$, $\sup_{x',x'' \in B(x,\delta)} |\chi_E(x') - \chi_E(x'')| = 1$. This implies that there exists $x', x'' \in B(x,\delta)$ s.t $\chi_E(x') = 1, \chi_E(x'') = 0$. (Exactly speaking, we can find a sequence $\{x'_n\}, \{x''_n\} \subset B(x,\delta)$ s.t $\chi_E(x'_n) - \chi_E(x''_n) \to 1$. For sufficiently large n, $\chi_E(x'_n) = 1, \chi_E(x''_n) = 0$. Therefore there exists $x', x'' \in B(x,\delta)$ s.t $\chi_E(x') = 1, \chi_E(x'') = 0$.) So $\forall \delta > 0$, $B(x,\delta) \cap E \neq \emptyset$ and $B(x,\delta) \cap E^c \neq \emptyset$. Hence $x \in \partial E = \overline{E} \setminus \mathring{E}$. Therefore $B = 1$.

Next, we show that $A \geqq B$. The proof is similar to the previous argument. Suppose that $B = 1$. Then $x \in \partial E = \overline{E} \setminus \mathring{E}$. So $\forall \delta > 0$, $B(x,\delta) \cap E \neq \emptyset$ and $B(x,\delta) \cap E^c \neq \emptyset$. We can find $x', x'' \in B(x,\delta)$ s.t $\chi_E(x') = 1$ and $\chi_E(x'') = 0$ for all $\delta > 0$. Therefore $\omega_f(x) = 1$. Now the proof is complete.

□

138 (Exercise 28) Let $g(x) \stackrel{\text{def}}{=} f(x^2)$. Since x^2 is continuous in $\mathbb{R}$, if $f(x)$ is continuous at $x_0 \in [0,1]$ then $g(x)$ is also continuous at x_0. This implies that

$$\begin{aligned} & D_g \stackrel{\text{def}}{=} \{x_0 \in [0,1] \mid g(x) \text{ is is discontinuous at } x_0\} \\ \subset\ & D_f \stackrel{\text{def}}{=} \{x_0 \in [0,1] \mid f(x) \text{ is is discontinuous at } x_0\} \end{aligned}$$

Therefore D_g is also countable. We conclude that g is also integrable on $[0,1]$. □

139 (Exercise 29) Since $f(x) + g(y) \in L(E \times E)$, by Fubini's Theorem,

$$\int_E (f(x) + g(y)) dx \in L(E) \text{ a.e } y \in E.$$

Therefore there exists $y_0 \in E$ s.t

$$\int_E (f(x) + g(y_0))dx \in L(E).$$

$g(y_0) \neq \pm\infty$ (otherwise, the integral above not integrable). So

$$\int_E (f(x) + g(y_0))dx = \int_E f(x)dx + m(E) \cdot g(y_0) \in (-\infty, \infty).$$

Therefore $\int_E f(x)dx \in (-\infty, \infty)$. Similarly $\int_E g(y)dy \in (-\infty, \infty)$. Now the proof is complete.

□

140 (Exercise 30) $\frac{1}{(1+y)(1+x^2y)}$ is non-negative so Tonelli's Theorem assures us that we may compute the integral as iterated integral or integral on $\mathbb{R}^2$.

STEP 1.

$$\begin{aligned}
&\int_{(x,y)\in(0,\infty)\times(0,\infty)} \frac{1}{(1+y)(1+x^2y)}dxdy \\
\overset{*1}{=}\ &\int_{y\in(0,\infty)} \left(\int_{x\in(0,\infty)} \frac{1}{(1+y)(1+x^2y)}dx\right) dy \\
\overset{*2}{=}\ &\int_{y\in(0,\infty)} \left(\lim_{c\to\infty} \int_{x\in(0,c)} \frac{1}{(1+y)(1+x^2y)}dx\right) dy \\
\overset{*3}{=}\ &\int_{y\in(0,\infty)} \left(\lim_{c\to\infty} \int_{t\in(0,\sqrt{y}c)} \frac{1}{(1+y)(1+t^2)} \frac{1}{\sqrt{y}}dt\right) dy \\
\overset{*4}{=}\ &\int_{y\in(0,\infty)} \left(\int_{t\in(0,\infty)} \frac{1}{(1+y)(1+t^2)} \frac{1}{\sqrt{y}}dt\right) dy \\
=\ &\left(\int_{y\in(0,\infty)} \frac{1}{(1+y)\sqrt{y}}dy\right) \left(\int_{t\in(0,\infty)} \frac{1}{(1+t^2)}dt\right)
\end{aligned}$$

- (∗1) Tonell's Theorem. We first compute $\int_{x\in(0,\infty)} \cdots dx$ and then $\int_{y\in(0,\infty)} \cdots dy$.
- (∗2) monotone convergence theorem. $\lim_{c\to\infty} \int_{(0,c)} \cdots = \lim_{c\to\infty} \int_{(0,\infty)} \chi_{(0,c)} \cdots = \int_{(0,\infty)} \lim_{c\to\infty} \chi_{(0,c)}$
- (∗3) $t = \sqrt{y}x$. §4.2. Example 10
- (∗4) monotone convergence theorem.

STEP 2. We use Riemann improper integral to compute the integral. (L) $\int_{t\in(0,\infty)} \frac{1}{(1+t^2)}dt =$ (L) $\int_{t\in[0,\infty)} \frac{1}{(1+t^2)}dt =$ (L) $\lim_{c\to\infty} \int_{t\in[0,c]} \frac{1}{(1+t^2)}dt$ by monotone convergence theorem. And $\int_{t\in[0,c]} \frac{1}{(1+t^2)}dt$ is Riemann integral. So we find

$$\text{(R)} \lim_{c\to\infty} \int_{t\in[0,c]} \frac{1}{(1+t^2)}dt = \frac{\pi}{2}.$$

Next, (L) $\int_{y\in(0,\infty)} \frac{1}{(1+y)\sqrt{y}}dy$ = (L) $\lim_{c_1\to+0,c_2\to\infty}\int_{y\in[c_1,c_2]} \frac{1}{(1+y)\sqrt{y}}dy$ by monotone convergece theorem. Similarly, $\int_{y\in[c_1,c_2]} \frac{1}{(1+y)\sqrt{y}}dy$ is Riemann integral so we find

$$\begin{aligned} &\text{(R)} \lim_{c_1\to+0,c_2\to\infty}\int_{y\in[c_1,c_2]} \frac{1}{(1+y)\sqrt{y}}dy \\ \overset{*}{=}\ &\text{(R)} \lim_{c_1\to+0,c_2\to\infty}\int_{z\in[\sqrt{c_1},\sqrt{c_2}]} \frac{2}{(1+z^2)}dz = \pi \end{aligned}$$

- ($*$) Let $z=\sqrt{y}$ and change variable.

Therefore the integral is $\frac{\pi^2}{2}$.

STEP 3.

$$\begin{aligned} &\int_{(x,y)\in(0,\infty)\times(0,\infty)} \frac{1}{(1+y)(1+x^2y)}dxdy \\ \overset{*5}{=}\ &\int_{x\in(0,\infty)} \left(\int_{y\in(0,\infty)} \frac{1}{(1+y)(1+x^2y)}dy\right)dx \\ \overset{*6}{=}\ &\int_{x\in(0,\infty)} \left(\lim_{c\to\infty}\int_{y\in[0,c]} \frac{1}{(1+y)(1+x^2y)}dy\right)dx \\ =\ &\int_{x\in(0,\infty)} \left(\lim_{c\to\infty}\int_{y\in[0,c]} \frac{1}{1-x^2}\left(\frac{1}{1+y}-\frac{x^2}{1+x^2y}\right)dy\right)dx \\ \overset{*7}{=}\ &\int_{x\in(0,\infty)} \left(\text{(R)}\lim_{c\to\infty}\int_{y\in[0,c]} \frac{1}{1-x^2}\left(\frac{1}{1+y}-\frac{x^2}{1+x^2y}\right)dy\right)dx \\ =\ &\int_{x\in(0,\infty)} \left(\lim_{c\to\infty}\frac{1}{1-x^2}\ln\left(\frac{1+c}{1+cx^2}\right)\right)dx \\ =\ &\int_{x\in(0,\infty)} \left(\frac{1}{1-x^2}\ln\left(\frac{1}{x^2}\right)\right)dx \\ =\ &\int_{x\in(0,\infty)} \left(\frac{2\ln(x)}{x^2-1}\right)dx \overset{*8}{=} \frac{\pi^2}{2} \end{aligned}$$

- ($*5$) Tonelli's Theorem.
- ($*6$) monotone convergence theorem
- ($*7$) $\frac{1}{1-x^2}\left(\frac{1}{1+y}-\frac{x^2}{1+x^2y}\right)$ is Riemann integrable on $y\in[0,c]$. So we find the integral as Riemann integral.
- ($*8$) by the result of Step 2.

□

141 (Exercise 31) $f(x)$ is a non-negative measurable function. (So we can apply Tonell's Theorem).

$$\begin{aligned}
\int_{\mathbb{R}} F(x)dx &= \int_{\mathbb{R}}\int_{E} f(x-t)dtdx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-t)\cdot\chi_E(t)\ dt\ dx \\
&\overset{*1}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-t)\cdot\chi_E(t)\ dx\ dt \\
&= \int_{\mathbb{R}} \chi_E(t)\cdot\int_{\mathbb{R}} f(x-t)\ dx\ dt \\
&\overset{*2}{=} \int_{\mathbb{R}} \chi_E(t)\cdot\int_{\mathbb{R}} f(x)\ dx\ dt \\
&= \int_{\mathbb{R}} \chi_E(t)\ dt\cdot\int_{\mathbb{R}} f(x)\ dx \\
&= m(E)\cdot\int_{\mathbb{R}} f(x)\ dx < \infty
\end{aligned}$$

- (∗1) Tonell's Theorem.
- (∗2) Translation does not change the value of integral. See Theorem 4.13.

Since $m(E) > 0$, $\int_{\mathbb{R}} f(x)\ dx < \infty$. Now the proof is complete. □

142 (Exercise 32) We show that both $\int_0^\infty F(x)dx$ and $\int_{-\infty}^0 F(x)dx$ are finite.

STEP 1. Since $xf(x) \in L(\mathbb{R})$, $\int_0^\infty xf(x)dx$ is finite.

$$\begin{aligned}
\int_0^\infty xf(x)dx &= \int_0^\infty\int_0^x f(x)dtdx \\
&\overset{*1}{=} \int_0^\infty\int_t^\infty f(x)dxdt \\
&\overset{*2}{=} \int_0^\infty\left(-\int_{-\infty}^t f(x)dx\right)dt \\
&= \int_0^\infty -F(t)\ dt \in \mathbb{R}
\end{aligned}$$

- (∗1) Fubini's Theorem.
- (∗2) By assumption $\int_{-\infty}^\infty f(x)dx = 0$ so $\int_{-\infty}^t f(x)dx + \int_t^\infty f(x)dx = 0$.

So $\int_0^\infty F(t)dt = \int_0^\infty(-xf(x))dx \in \mathbb{R}$.

STEP 2. Repeat a similar argument on $\int_{-\infty}^{0} -xf(x)dx$.

$$\begin{aligned}\int_{-\infty}^{0} -xf(x)dx &= \int_{-\infty}^{0}\int_{x}^{0} f(x)dtdx \\ &\overset{*3}{=} \int_{-\infty}^{0}\int_{-\infty}^{t} f(x)dxdt \\ &= \int_{-\infty}^{0} F(t)dt \in \mathbb{R}\end{aligned}$$

- (∗3) Fubini's Theorem.

By merging these two conclusions, we have $\int_{-\infty}^{\infty}(-xf(x))dx = \int_{-\infty}^{\infty} F(t)dt \in \mathbb{R}$.

□

143 (Exercise 33) We apply Lebesgue Dominated Convergence Theorem.

$$|\cos x \arctan nx| \leqq \frac{\pi}{2}\cos x \in L([0, \frac{\pi}{2}]).$$

So

$$\begin{aligned}\lim_{n\to\infty}\int_0^{\frac{\pi}{2}} \cos x \arctan(nx)dx &= \int_0^{\frac{\pi}{2}} \lim_{n\to\infty} \cos x \arctan(nx)dx \\ &\overset{*}{=} \int_0^{\frac{\pi}{2}} \cos x \cdot \frac{\pi}{2}dx \\ &= \frac{\pi}{2}\ (\mathrm{R})\int_0^{\frac{\pi}{2}} \cos x dx = \frac{\pi}{2}\end{aligned}$$

- (∗) if $x \in (0, \frac{\pi}{2}]$, $\arctan nx \to \frac{\pi}{2}$. So $\arctan nx \to \frac{\pi}{2}$ a.e $x \in [0, \frac{\pi}{2}]$

□

144 (Exercise 34)

STEP 1. $(g \in L(I))$

$$\begin{aligned}\int_0^a |g(x)|\,dx &= \int_0^a \left|\int_x^a \frac{f(t)}{t}\ dt\right| dx \\ &\overset{*1}{\leqq} \int_0^a\int_x^a \left|\frac{f(t)}{t}\right|\ dt\ dx \\ &\overset{*2}{=} \int_0^a\int_0^t \frac{|f(t)|}{t}\ dx\ dt \\ &= \int_0^a \frac{|f(t)|}{t}\cdot t\ dt \\ &= \int_0^a |f(t)|dt < \infty\end{aligned}$$

- ($*1$) triangular inequality
- ($*2$) Tonelli's Theorem

STEP 2. ($\int_I g = \int_I f$)

$$\begin{aligned}\int_0^a g(x)\ dx &= \int_0^a \int_x^a \frac{f(t)}{t}\ dt\ dx \\ &\overset{*3}{=} \int_0^a \int_0^t \frac{f(t)}{t} dx\ dt \\ &= \int_0^a \int_0^t \frac{f(t)}{t} \cdot t\ dt \\ &= \int_0^a f(t)\ dt\end{aligned}$$

- ($*3$) Fubini's Theorem. We already know that $g(x) \in L(I)$, so we can swap dt and dx.

□

CHAPTER 5

Solutions

§ 5.1

1 (Definition 5.1) For all $x \in E$ and for all $\epsilon > 0$, there exists $I \in \Gamma$ s.t $x \in I$ and $\operatorname{diam}(I) < \epsilon$, then we say that Γ is a Vitalli cover of E.

- Until now, $|\cdot|$ is defined for open intervals. (i.e if $I = \prod_{j=1}^{d}(a_i, b_i)$, then $|I| \stackrel{\text{def}}{=} \prod_{j=1}^{d}(b_i - a_i)$.) However, we extend the definition of $|\cdot|$ to closed intervals and half-open intervals.
- Note that $\operatorname{diam}(I) = |I|$ when $E \subset \mathbb{R}^1$.

□

2 (Example 1) Let $\{r_m\}_{m \geqq 1} \stackrel{\text{def}}{=} \mathbb{Q} \cap [a,b]$. Let $\Gamma \stackrel{\text{def}}{=} \{I_{m,n}\}_{m\in\mathbb{N}, n\in\mathbb{N}}$ where $I_{m,n} \stackrel{\text{def}}{=} [r_m - \frac{1}{n}, r_m + \frac{1}{n}]$. We claim that Γ is a Vitalli cover of $[a,b]$. We pick $n \in \mathbb{N}$ s.t $\frac{2}{n} < \epsilon$. For every $x \in [a,b]$ we can find $r_m \in \mathbb{Q} \cap [a,b]$ s.t $|x - r_m| < \frac{1}{n}$. ($\mathbb{Q}$ is dense.) So $x \in I_{m,n}$ and $\operatorname{diam}(I_{m,n}) = \frac{2}{n} < \epsilon$. □

3 (Theorem 5.1 Vitalli's Covering Lemma) We pick $G \in \mathscr{O}^1$ (an open set) s.t $E \subset G$ with $m(G) < \infty$. (We can find such G because $m^*(E) < \infty$. Let us consider $\{J_n\}_{n\geqq 1}$ s.t $E \subset \bigcup_{n=1}^{\infty} J_n$ with $\sum_{n=1}^{\infty} |J_n| < m^*(E) + 1 < \infty$. Let $G \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} J_n$.)

We may suppose that $\forall I \in \Gamma$, $I \subset G$ without loss of generality. Let $x \in E$. Then $x \in G$. There exists $\delta > 0$ s.t $B(x, \delta) \subset G$. Since Γ is a Vitalli cover, we can find $I \in \Gamma$ s.t $x \in I$ and $\operatorname{diam}(I) < \delta$. Then $I \subset G$. So we suppose that every $I \in \Gamma$ is contained in G.

STEP 1. We pick an arbitrary interval from $I_1 \in \Gamma$. Now suppose that we have chosen $\{I_1, \cdots, I_k\} \subset \Gamma$ ($k \geqq 1$). If $E \subset \bigcup_{j=1}^{k} I_j$, then the statement holds obviously, and we do not have to prove anymore. So we we suppose $E \not\subset \bigcup_{j=1}^{k} I_j$ for all $k \geqq 1$. Let us define

$$\delta_k \stackrel{\text{def}}{=} \sup\{|I| \mid I \in \Gamma \text{ with } I \cap I_j = \emptyset \text{ for all } j = 1, \cdots, k\}.$$

Note that $\delta_k < \infty$ because $I \subset G$ for all $I \in \Gamma$ by our assumption. We can find $I_{k+1} \in \Gamma$ s.t

$$|I_{k+1}| > \frac{1}{2}\delta_k \text{ and } I_{k+1} \cap I_j = \emptyset \text{ for all } j = 1, \cdots, k.$$

Since $\{I_n\}_{n \geqq 1}$ are disjoint with each other and $\bigcup_{n=1}^{\infty} I_n \subset G$, we have

$$\begin{aligned} m\left(\bigcup_{n=1}^{\infty} I_n\right) &\overset{*1}{=} \sum_{n=1}^{\infty} |I_n| \\ &\leqq m(G) < \infty \end{aligned}$$

- $(*1)$ $m(I_n) = |I_n|$. (See §2.1). So $m\left(\bigcup_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{\infty} |I_n|$.

Since $\sum_{j=1}^{n} |I_n| \to \sum_{n=1}^{\infty} |I_n| < \infty$ as $n \to \infty$, we can find sufficiently large $n \in \mathbb{N}$ s.t.

$$\sum_{j=n+1}^{\infty} |I_j| < \frac{\epsilon}{5}.$$

Let

$$S \overset{\text{def}}{=} E \setminus \bigcup_{j=1}^{n} I_j.$$

Our goal is to prove that

$$m^*(S) < \epsilon.$$

STEP 2. Let I_j^* $(j = 1, 2, \cdots)$ be the interval which has the common center with I_j and whose edge length is 5 times I_j. It is enough for us to prove that

$$S \subset \bigcup_{j=n+1}^{\infty} I_j^*,$$

because

$$m^*(S) \leqq \sum_{j=n+1}^{\infty} m(I_j^*) = \sum_{j=n+1}^{\infty} |I_j^*| = 5 \sum_{j=n+1}^{\infty} |I_j| < 5 \cdot \frac{\epsilon}{5} = \epsilon,$$

and so the proof is complete.

STEP 3. We prove that $S \subset \bigcup_{j=n+1}^{\infty} I_j^*$. We pick an arbitrary point $x \in S$ and show that there always exists sufficiently large $n_0 \in \mathbb{N}$ s.t $x \in I_{n_0}^*$. By our assumption that $\{I_j\}_{j \geqq 1}$ are closed intervals, $F \overset{\text{def}}{=} \bigcup_{j=1}^{n} I_j$ is a closed set. Let us recall that $\delta_x \overset{\text{def}}{=} \text{dist}(x, F) = |x - y|$ for some $y \in F$ by Theorem 1.24. Since $x \notin F$, $\text{dist}(x, F) > 0$ (otherwise $|x - y| = 0 \Leftrightarrow x = y$ for some $y \in F$), $\delta_x > 0$. Since Γ is a Vitalli cover, we can find $I_x \in \Gamma$ with $\text{diam}(I_x) < \delta_x$. Then I_x and $F = \bigcup_{j=1}^{n} I_j$ are disjoint. So $I_x \cap I_j = \emptyset$ for all $j = 1, \cdots, n$.

We claim that there exists sufficiently large $n_0 > n$ s.t $I_x \cap I_{n_0} \neq \emptyset$. To prove this, suppose that $I_x \cap I_j = \emptyset$ for all $j = 1, 2, \cdots$. Note that $|I_j| = \text{diam}(I_j) \to 0$ as $j \to \infty$ because $\sum_{j=1}^{\infty} |I_j| < \infty$. So we can find sufficiently large $j_0 \in \mathbb{N}$ s.t

$$|I_{j_0+1}| < \frac{1}{2}|I_x|. \ (*2)$$

Let us recall that

$$\delta_{j_0} \stackrel{\text{def}}{=} \sup\{|I| \mid I \cap I_j = \emptyset \text{ for all } j = 1, \cdots, j_0\}.$$

Since we suppose that $I_x \cap I_j$ for all $j = 1, 2, \cdots$, (so $|I_x| \in \{|I| \mid I \in \Gamma \text{ with } I \cap I_j \text{ for all } j = 1, \cdots, j_0\}$), we have

$$|I_x| \leqq \delta_{j_0}. \ (*3)$$

By merging these two results $(*2, 3)$, we obtain

$$|I_{j_0+1}| < \frac{1}{2}|I_x| \leqq \frac{1}{2}\delta_{j_0}. \ (*4)$$

However we chose $\{I_j\}_{j \geqq 1}$ so that

$$|I_{j_0+1}| > \frac{1}{2}\delta_{j_0} \ (*5)$$

in STEP 1. And $(*4)$ and $(*5)$ contradicts to each other. So we conclude that there exists $n_0 > n$ s.t $I_x \cap I_{n_0} \neq \emptyset$.

STEP 4. We suppose that n_0 is the smallest index s.t $I_x \cap I_{n_0} \neq \emptyset$. So $I_x \cap I_j = \emptyset$ for $j = 1, \cdots, n_0 - 1$. Therefore

$$|I_x| \leqq \delta_{n_0-1} \stackrel{\text{def}}{=} \sup\{|I| \mid I \in \Gamma \text{ with } I \cap I_j = \emptyset \text{ for all } j = 1, \cdots, n_0 - 1\}.$$

Let us recall that we chose $\{I_j\}_{j \geqq 1}$ s.t

$$|I_{n_0}| > \frac{1}{2}\delta_{n_0-1}.$$

By merging these two results we have,

$$|I_x| < 2|I_{n_0}|.$$

Since $x \in I_x$, $I_x \cap I_{n_0} \neq \emptyset$ (not disjoint) and $\text{diam}(I_x) = |I_x|$ is less than twice $\text{diam}(I_{n_0}) = |I_{n_0}|$,

$$x \in I_x \subset I_{n_0}^*,$$

where $I_{n_0}^*$ is the interval which has the common center with I_{n_0} and whose edge length is 5 times I_{n_0}. (You may draw a figure to see this fact.) In conclusion, $\forall x \in S$, there exists $n_0 > n$ s.t $x \in I_{n_0}^*$. Therefore

$$S \subset \bigcup_{j=n+1}^{\infty} I_j^*.$$

Now the proof is complete.

□

4 (Definition 5.2)

$$\begin{aligned}
D^+f(x_0) &\overset{\text{def}}{=} \limsup_{h\to+0} \frac{f(x_0+h)-f(x_0)}{h} \\
&= \lim_{h\to+0} \sup_{k\in(0,h)} \frac{f(x_0+k)-f(x_0)}{k} \\
D_+f(x_0) &\overset{\text{def}}{=} \liminf_{h\to+0} \frac{f(x_0+h)-f(x_0)}{h} \\
&= \lim_{h\to+0} \inf_{k\in(0,h)} \frac{f(x_0+k)-f(x_0)}{k} \\
D^-f(x_0) &\overset{\text{def}}{=} \limsup_{h\to-0} \frac{f(x_0+h)-f(x_0)}{h} \\
&= \lim_{h\to-0} \sup_{k\in(h,0)} \frac{f(x_0+k)-f(x_0)}{k} \\
D_-f(x_0) &\overset{\text{def}}{=} \liminf_{h\to-0} \frac{f(x_0+h)-f(x_0)}{h} \\
&= \lim_{h\to-0} \inf_{k\in(h,0)} \frac{f(x_0+k)-f(x_0)}{k}
\end{aligned}$$

If $D^+f(x_0) = D_+f(x_0) = D^-f(x_0) = D_-f(x_0)$, then we say that $f(x)$ is differentiable at $x = x_0$. Note that

$$D^+f(x_0) \geqq D_+f(x_0),$$

and

$$D^-f(x_0) \geqq D_-f(x_0)$$

always holds.

□

5 (Theorem 5.2 Lebesgue's Theorem)

(1) We show that

$$D^+f(x) = D_+f(x) = D^-f(x) = D_-f(x) \text{ a.e } x \in [a,b].$$

Let

$$\begin{aligned}
E_1 &\overset{\text{def}}{=} \{x \in [a,b] \mid D^+f(x) > D_-f(x)\} \\
E_2 &\overset{\text{def}}{=} \{x \in [a,b] \mid D^-f(x) > D_+f(x)\}.
\end{aligned}$$

We show that

$$m(E_1) = m(E_2) = 0.$$

Then it follows that

$$D^+f(x) \leqq D_-f(x) \overset{*1}{\leqq} D^-f(x) \leqq D_+f(x) \overset{*2}{\leqq} D^+f(x) \text{ a.e } x \in [a,b].$$

- $(*1, 2)$ These inequality always hold by definition. $\liminf \leqq \limsup$.

And we have the desired conclusion. Let $g(x) = -f(x)$. Then $g(x)$ is a monotone decreasing function on $[a, b]$, and

$$E_2 = \{x \in [a, b] \mid D^+ g(x) > D_- g(x)\},$$

because

$$D^+ g = D^+(-f) = -D_+ f \text{ and } D_- g = D_-(-f) = -D^- f.$$

And the proofs of $m(E_1) = 0$ and $m(E_2) = 0$ are quite similar. (monotone increasing vs monotone decreasing) It is sufficient for us to show that $m(E_1)$.

Let $\mathbb{Q}^+ \overset{\text{def}}{=} \mathbb{Q} \cap (0, \infty)$. Note that

$$E_1 = \bigcup_{r,s \in \mathbb{Q}^+} \{x \in [a, b] \mid D^+ f(x) > r > s > D_- f(x)\}.$$

Note that $D_- f(x) \geqq 0$ because $f(x)$ is monotone increasing. So it is sufficient to pick $r > s \in \mathbb{Q}^+$ (but not $\mathbb{Q}$) in the equality above. Let

$$A_{r,s} \overset{\text{def}}{=} \{x \in [a, b] \mid D^+ f(x) > r > s > D_- f(x)\}.$$

It is sufficient for us to show that

$$m(A_{r,s}) = 0,$$

for each $(r, s) \in \mathbb{Q}^+ \times \mathbb{Q}^+$ $(r > s)$ because

$$m(E_1) \leqq \sum_{r,s \in \mathbb{Q}^+} m(A_{r,s}).$$

Now we fix $r, s \in \mathbb{Q}^+$ and let

$$A \overset{\text{def}}{=} A_{r,s}.$$

STEP 1. Let $\epsilon > 0$ be an arbitrary positiver number. Let G be an open set with $G \supset A$ and

$$m(G) < (1 + \epsilon) m^*(A). \ (*a)$$

Actually $A \in \mathscr{M}$, however we can derive the result without the assumption that $A \in \mathscr{M}$. So we use $m^*(A)$ instead of $m(A)$. For every $x \in A$, since

$$\begin{aligned} D_- f(x) &= \liminf_{h \to -0} \frac{f(x+h) - f(x)}{h} \\ &= \liminf_{h \to +0} \frac{f(x-h) - f(x)}{-h} \\ &= \lim_{k \to +0} \inf_{h \in (0,k)} \frac{f(x-h) - f(x)}{-h} < s, \end{aligned}$$

we have

$$\inf_{h \in (0,k)} \frac{f(x-h) - f(x)}{-h} < s,$$

for every $k > 0$ (especially for arbitrarily small $k > 0$). And we can find $h \in (0, k)$ s.t

$$\frac{f(x-h) - f(x)}{-h} < s.$$

Since $x \in A \subset G$ and G is an open set, when $h > 0$ is sufficiently small,

$$[x-h, x] \subset G.$$

In conclusion, for every $x \in A$ and for every $\delta > 0$, we can find $h \in (0, \delta)$

$$\frac{f(x-h) - f(x)}{-h} < s \text{ and } [x-h, x] \subset G$$

Therefore such $\{[x-h, x]\}$ is a Vitalli cover of A. By Theorem 5.1 Vitalli's Covering Theorem, there exist disjoint closed intervals $\{[x_j - h_j, x_j]\}_{j=1}^p$ s.t

$$m^* \left(A \setminus \bigcup_{j=1}^p [x_j - h_j, x_j] \right) < \epsilon. \ (*b)$$

Note that $\bigcup_{j=1}^p [x_j - h_j, h_j]$ is a Lebesgue measurable set. By definition of Lebesgue measurability, for all $A \subset \mathbb{R}$ we have

$$m^*(A) = m^* \left(A \cap \bigcup_{j=1}^p [x_j - h_j, x_j] \right) + m^* \left(A \setminus \bigcup_{j=1}^p [x_j - h_j, x_j] \right). \ (*c)$$

By $(*b)$ and $(*c)$, we have

$$m^* \left(A \cap \bigcup_{j=1}^p [x_j - h_j, x_j] \right) > m^*(A) - \epsilon.$$

STEP 2. Note that

$$\frac{f(x_j - h_i) - f(x_j)}{-h_j} < s \qquad (\Leftrightarrow f(x_j) - f(x_j - h_j) < sh_j).$$

So we have

$$\sum_{j=1}^p (f(x_j) - f(x_j - h_j)) < s \sum_{j=1}^p h_j \overset{*2.1}{<} s(1+\epsilon) m^*(A).$$

We explain $(*2.1)$.

$$\sum_{j=1}^p h_j \overset{*2.2}{=} m \left(\bigcup_{j=1}^p [x_j - h_j, x_j] \right) \overset{*2.3}{\leqq} m(G) \overset{*2.4}{<} (1+\epsilon) m^*(A).$$

- $(*2.2)$ $\{[x_j - h_j, x_j]\}_{j=1}^p$ are disjoint.
- $(*2.3)$ $\bigcup_{j=1}^p [x_j - h_j, x_j] \subset G$.
- $(*2.4)$ See $(*a)$

STEP 3. Let

$$B \overset{\text{def}}{=} A \cap \bigcup_{j=1}^{p} (x_j - h_j, x_j).$$

We repeat a similar argument. For every $y \in B$, $D^+ f(y) > r$. Note that

$$\begin{aligned} D^+ f(y) &= \limsup_{h \to +0} \frac{f(y+h) - f(y)}{h} \\ &= \lim_{h \to +0} \sup_{k \in (0,h)} \frac{f(y+k) - f(y)}{k} > r. \end{aligned}$$

So for every $h > 0$ (especially for arbitrarily small $h > 0$),

$$\sup_{k \in (0,h)} \frac{f(y+k) - f(y)}{k} > r,$$

hence we can find $k \in (0, h)$ s.t

$$\frac{f(y+k) - f(y)}{k} > r.$$

By taking sufficiently small $k > 0$, we can satisfy

$$[y, y+k] \subset (x_j - h_j, x_j) \text{ for some } j = 1, \cdots, p,$$

because $y \in B \subset \bigcup_{j=1}^{p} (x_j - h_j, x_j)$, and each $(x_j - h_j, x_j)$ is open. In conclusion, for every $y \in B$ and for every $\delta > 0$, we can find $k \in (0, \delta)$ s.t

$$\frac{f(y+k) - f(y)}{k} > r \text{ and } [y, y+k] \subset (x_j - h_j, x_j) \text{ for some } j = 1, \cdots, p.$$

Therefore, such $\{[y, y+k]\}_{y,k}$ is a Vitalli cover. By Theorem 5.1 Vitalli's Covering Theorem, we can find disjoint closed intervals $\{[y_i, y_i + k_i]\}_{i=1}^{q}$ s.t

$$m^* \left(B \setminus \bigcup_{i=1}^{q} [y_i, y_i + k_i] \right) < \epsilon.$$

Therefore

$$\begin{aligned}
\sum_{i=1}^{q} k_i &= \sum_{i=1}^{q} m([y_i, y_i + k_i]) \\
&\overset{*3.1}{=} m\left(\bigcup_{i=1}^{q}[y_i, y_i + k_i]\right) \\
&\geqq m^*\left(B \cap \bigcup_{i=1}^{q}[y_i, y_i + k_i]\right) \\
&\overset{*3.2}{=} m^*(B) - m^*\left(B \setminus \bigcup_{i=1}^{q}[y_i, y_i + k_i]\right) \\
&> m^*(B) - \epsilon \\
&= m^*\left(A \cap \bigcup_{j=1}^{p}(x_j - h_j, x_j)\right) - \epsilon \\
&\overset{*3.3}{=} m^*\left(A \cap \bigcup_{j=1}^{p}[x_j - h_j, x_j]\right) - \epsilon \\
&\overset{*3.4}{>} m^*(A) - \epsilon - \epsilon
\end{aligned}$$

- $(*3.1)$ $\{[y_i, y_i + k_i]\}_{i=1}^{q}$ are disjoint.
- $(*3.2)$ Since $\bigcup_{i=1}^{q}[y_i, y_i + k_i] \in \mathscr{M}$, we have $m^*(B) = m^*\left(B \cap \bigcup_{i=1}^{q}[y_i, y_i + k_i]\right) + m^*\left(B \setminus \bigcup_{i=1}^{q}[y_i, y_i + k_i]\right)$.
- $(*3.3)$ Use sub-additivity of Lebesgue measure. Then recall that a countable set is a measure zero set.

$$\begin{aligned}
m^*\left(A \cap \bigcup_{j=1}^{p}(x_j - h_j, x_j)\right) &\leqq m^*\left(A \cap \bigcup_{j=1}^{p}[x_j - h_j, x_j]\right) \\
&\leqq m^*\left(A \cap \bigcup_{j=1}^{p}(x_j - h_j, x_j)\right) + m^*\left(A \cap \bigcup_{j=1}^{p}\{x_j - h_j, x_j\}\right) \\
&\leqq m^*\left(A \cap \bigcup_{j=1}^{p}(x_j - h_j, x_j)\right) + m^*\left(\bigcup_{j=1}^{p}\{x_j - h_j, x_j\}\right)
\end{aligned}$$

- $(*3.4)$ By the conclusion of STEP1.

Furthermore, for each $i = 1, \cdots, q$, we have

$$\frac{f(y_i + k + i) - f(y_i)}{k_i} > r \qquad (\Leftrightarrow f(y_i + k_i) - f(y_i) > rk_i).$$

So

$$\sum_{i=1}^{q}(f(y_i + k_i) - f(y_i)) > r\sum_{i=1}^{q} k_i > r(m^*(A) - 2\epsilon)$$

STEP 4. Let us recall that for each $i = 1, \cdots, q$, there exists j s.t $[y_i, y_i + k_i] \subset (x_j - h_j, x_j)$. Furthermore, $f(x)$ is a monotone increasing function on $[a, b]$. These facts imply that

$$\sum_{i=1}^{q} f(y_i + k_i) - f(y_i) \leqq \sum_{j=1}^{p} f(x_j) - f(x_j - h_j)$$

By the conclusion of STEP2 and STEP3, we have

$$r(m^*(A) - 2\epsilon) < s(1 + \epsilon)m^*(A).$$

By taking $\epsilon \to +0$, we have

$$rm^*(A) \leqq sm^*(A).$$

Since $r > s$, we conclude that

$$m^*(A) = 0.$$

Now the proof is complete.

(2) Let

$$f_n(x) \overset{\text{def}}{=} n\left(f_n\left(x + \frac{1}{n}\right) - f(x)\right).$$

Note that $f_n(x)$ is a non-negative measurable function define on $[a, b]$. We may suppose that

$$f(x) = f(b) \qquad (\text{if } x > b).$$

Since $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists a.e $x \in [a, b]$, $\lim_{n\to\infty} f_n(x)$ exists a.e $x \in [a, b]$. And we define

$$f'(x) \overset{\text{def}}{=} \begin{cases} \lim_{h\to\infty} \frac{f(x+h)-f(x)}{h} & \text{if exists} \\ 0 & \text{otherwise} \end{cases}.$$

Since $\lim_{h\to\infty} \frac{f(x+h)-f(x)}{h}$ does not always exist, we modify the definition of $f'(x)$ so that $f'(x)$ becomes a measurable function defined everywhere on $[a, b]$. (Note that the modification is only done on a measure zero set, so it does not have an influence on the integral.) However, some people do not implement the modification above, and directly treat $f'(x) \overset{\text{def}}{=} \lim_{h\to\infty} \frac{f(x+h)-f(x)}{h}$ as a measurable function defined a.e $x \in [a, b]$.

Anyway now $f'(x)$ is a measurable function, and $f'(x) \geqq 0$ because $f(x)$ is monotone increasing on $[a, b]$. Furthermore, $\lim_{n\to\infty} f_n(x) = f'(x)$ a.e $x \in [a, b]$. By applying Fatou's lemma to $\{f_n(x)\}_{n \geqq 1}$, we have

$$
\begin{aligned}
\int_{[a,b]} f'(x)dx &\overset{*1}{=} \int_{[a,b]} \liminf_{n\to\infty} f_n(x)dx \\
&\overset{*2}{\leqq} \liminf_{n\to\infty} \int_{[a,b]} f_n(x)dx \\
&\overset{*3}{=} \liminf_{n\to\infty} \int_{[a,b]} n\left(f\left(x+\frac{1}{n}\right) - f(x)\right) dx \\
&\overset{*4}{=} \liminf_{n\to\infty} \left(\int_{[a,b]} nf\left(x+\frac{1}{n}\right) dx - \int_{[a,b]} nf(x)dx\right) \\
&\overset{*5}{=} \liminf_{n\to\infty} \left(n\int_{[a,b]} f\left(x+\frac{1}{n}\right) dx - n\int_{[a,b]} f(x)dx\right) \\
&\overset{*6}{=} \liminf_{n\to\infty} \left(n\int_{[a+1/n,b+1/n]} f\left(x\right) dx - n\int_{[a,b]} f(x)dx\right) \\
&\overset{*7}{=} \liminf_{n\to\infty} \left(n\int_{[b,b+1/n]} f\left(x\right) dx - n\int_{[a,a+1/n]} f(x)dx\right) \\
&\overset{*8}{=} \liminf_{n\to\infty} \left(n\int_{[b,b+1/n]} f(b)dx - n\int_{[a,a+1/n]} f(x)dx\right) \\
&\overset{*9}{\leqq} \liminf_{n\to\infty} \left(n\int_{[b,b+1/n]} f(b)dx - n\int_{[a,a+1/n]} f(a)dx\right) \\
&= \liminf_{n\to\infty} \left(f(b) - f(a)\right)
\end{aligned}
$$

- (∗1) $f'(x) = \liminf_{n\to\infty} f_n(x)$ a.e $x \in [a,b]$.
- (∗2) Fatou's lemma.
- (∗3) By definition.
- (∗4) Note that $f(x)$ is intregrable on $[a,b]$. Note that $|f(x)| \leqq \max\{|f(a)|, |f(b)|\} < \infty$ ($f(x)$ is a real-valued function), and $[a,b]$ is bounded.
- (∗5) Put n outside the integral. (Theorem 4.10)
- (∗6) Rewrite $\int_{\mathbb{R}} f(x+\frac{1}{n})\chi_{[a,b]}(x)dx$. Then apply Theorem 4.13.
- (∗7) Simple rearrangement.
- (∗8) $f(x) = b$ when $x > b$.
- (∗9) $f(x)$ is monotone increasing, so $f(x) \geqq f(a)$.

□

6 (Theorem 5.3) Let

$$S(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} f_n(x), \quad S_n(x) \overset{\text{def}}{=} \sum_{k=1}^{n} f_k(x), \quad R_n(x) \overset{\text{def}}{=} \sum_{k=n+1}^{\infty} f_k(x).$$

Since $S(x)$ converges (is well-defined and is finite), $R(x)$ also converges. By assumption, each $f_n(x)$ is monotone increasing on $[a,b]$, so $f_n'(x)$ exists a.e $x \in [a,b]$. (Theorem 5.2) Let A_n be a measure zero set where $f_n'(x)$ exists $x \in [a,b] \setminus A_n$. Since $A \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} A_n$ is also a measure zero set, we can say that $f_n'(x)$ exists for all $n \in \mathbb{N}$ a.e $x \in [a,b]$. Note that $S(x), S_n(x), R_n(x)$ are also monotone increasing function on $[a,b]$. Simiarly, $S'(x), S_n'(x), R_n'(x)$ exists for all $n \in N$ a.e $x \in [a,b]$.

STEP 1. Note that

$$S(x) = S_n(x) + R_n(x),$$

because

$$\begin{aligned} S(x) &= \lim_{k\to\infty} S_k(x) \\ &= \lim_{k\to\infty} \left(\sum_{k=1}^{n} f_k(x) + \sum_{j=n+1}^{k} f_j(x) \right) \\ &= S_n(x) + \lim_{k\to\infty} \sum_{j=n+1}^{k} f_j(x) \\ &= S_n(x) + R_n(x). \end{aligned}$$

From the previous discussion, $S'(x), S_n'(x), R_n'(x)$ exists a.e $x \in [a,b]$. So we have

$$S'(x) = S_n'(x) + R_n'(x) \text{ a.e } x \in [a,b].$$

for each $n \in \mathbb{N}$. Note that

$$S_n'(x) = \frac{d}{dx} S(x) = \frac{d}{dx} \sum_{k=1}^{n} f_k(x) \overset{*1}{=} \sum_{k=1}^{n} \frac{d}{dx} f_k(x) \text{ a.e } x \in [a,b].$$

- ($*1$) Recall that $(f+g)' = f' + g'$ if f', g' exists. (For a sum of a finite number of differentiable functions, we can swap $\sum$ and $\frac{d}{dx}$. In this theorem, we prove that we can swap $\sum$ and $\frac{d}{dx}$ for a sum of a countably infinite number of differentiable functions.)

So we have

$$S'(x) = \sum_{k=1}^{n} f_k'(x) + R_n'(x) \text{ a.e } x \in [a,b],$$

for each $n \in \mathbb{N}$. All we have to do is to prove that

$$\lim_{n\to\infty} R_n'(x) = 0 \text{ a.e } x \in [a,b],$$

STEP 2. Note that

$$R_n(x) = f_{n+1}(x) + R_{n+1}(x),$$

because

$$R_n(x) = \lim_{k\to\infty}\left(\sum_{j=n+2}^{k} f_j(x) + f_{n+1}(x)\right) = \sum_{j=n+2}^{\infty} f_j(x) + f_{n+1}(x).$$

Recall that $R_n'(x), f_n'(x)$ exist for all $n \in \mathbb{N}$ a.e $x \in [a,b]$. So we have

$$R_n'(x) = f_{n+1}'(x) + R_{n+1}'(x) \text{ a.e } x \in [a,b].$$

Since $f_{n+1}'(x) \geqq 0$ (if exists), we have

$$R_n'(x) \geqq R_{n+1}'(x) \text{ a.e } x \in [a,b].$$

So $R_n'(x)$ exists a.e $x \in [a,b]$ and $\{R_n'(x)\}$ is a decreasing sequence with respect to $n \in \mathbb{N}$. This implies that

$$\lim_{n\to\infty} R_n'(x) \text{ exists a.e } x \in [a,b].$$

We define

$$R^*(x) \overset{\text{def}}{=} \begin{cases} \lim_{n\to\infty} R_n'(x) & \text{if exists} \\ 0 & \text{otherwise} \end{cases}.$$

Note that $R_n'(x) \geqq 0$ (if exists), so $R^*(x) \geqq 0$. Therefore $R^*(x)$ is a non-negative measurable function define on $[a,b]$.

STEP 3. By Fatou's lemma (we apply to $R_n(x)$) and Theorem 5.2

$$\begin{aligned}
\int_{[a,b]} R^*(x)dx &\overset{*2}{=} \int_{[a,b]} \liminf_{n\to\infty} R_n'(x)dx \\
&\overset{*3}{\leqq} \liminf_{n\to\infty} \int_{[a,b]} R_n'(x)dx \\
&\overset{*4}{\leqq} \liminf_{n\to\infty}(R_n(b) - R_n(a)) \\
&\overset{*5}{=} 0.
\end{aligned}$$

- $(*2)$ Since $\lim_{n\to\infty} R_n'(x)$ exists a.e $x \in [a,b]$, $R^*(x) = \liminf_{n\to\infty} R_n'(x)$ a.e $x \in [a,b]$.
- $(*3)$ Fatou's lemma.
- $(*4)$ Theorem 5.2.
- $(*5)$ Recall that $R_n(a) = \sum_{k=n+1}^{\infty} f_k(a)$ converges. (exists and is finite) So when $n \to \infty$, $R_n(a) \to 0$. (Basic calculus) Similarly $R_n(b) \to 0$ as $n \to \infty$.

This implies that $R^*(x) = 0$ a.e $x \in [a,b]$. Therefore if $\lim_{n\to\infty} R_n'(x)$ exists, then $\lim_{n\to\infty} R_n'(x) = 0$ a.e $x \in [a,b]$. (So we can say that $\lim_{n\to\infty} R_n'(x) = 0$ a.e $x \in [a,b]$.) Now the proof is complete.

□

7 (Exercise 1) Suppose that $F(x)$ is a real-valued primitive function of $f(x)$. Since $F'(x) = f(x) \geqq 0$ ($f(x)$ is non-negative by assumption), $F(x)$ is monotone-increasing. By Theorem 5.2, we have

$$\int_{[a,b]} f(x)dx = \int_{[a,b]} F'(x)dx \leqq F(b) - F(a) \in [0, \infty).$$

So $f(x)$ is integrable on $[a, b]$. This contradicts to the assumption. Now the proof is compelte. □

8 (Exercise 2)

STEP 1. Since $\lim_{n\to\infty} f_n(x) = 1$ a.e $x \in (0, 1)$, we can pick $b_k \nearrow 1$ s.t

$$\lim_{n\to\infty} f_n(b_k) = 1 \text{ for all } k = 1, 2, \cdots.$$

Otherwise, there exists some $b \in (0, 1)$ s.t $\forall x \in [b, 1)$ $\lim_{n\to\infty} f_n(x) \neq 1$. Similarly, we can pick $a_k \searrow 0$ s.t

$$\lim_{n\to\infty} f_n(a_k) = 1 \text{ for all } k = 1, 2, \cdots.$$

STEP 2. By Theorem 5.2, $f_n'(x)$ exists a.e $x \in (0, 1)$ and $f_n'(x) \geqq 0$ if exists. Virtually we can regard $f_n'(x)$ as a non-negative measurable function. (If $f_n'(x)$ does not exist, then we assume $f_n'(x) = 0$.) By applying Fatou's Lemma and Theorem 5.2, we have

$$\begin{aligned} 0 \leqq \int_{[a_k,b_k]} \liminf_{n\to\infty} f_n'(x)dx &\overset{*1}{\leqq} \liminf_{n\to\infty} \int_{[a_k,b_k]} f_n'(x)dx \\ &\overset{*2}{\leqq} \liminf_{n\to\infty} (f_n(b_k) - f_n(a_k)) \overset{*3}{=} 1 - 1 = 0. \end{aligned}$$

- $(*1)$ Fatou's Lemma.
- $(*2)$ Theorem 5.2.
- $(*3)$ $\lim_{n\to\infty} f_n(a_k) = 1$ and $\lim_{n\to\infty} f_n(b_k) = 1$.

So for every $k \in \mathbb{N}$,

$$\int_{[a_k,b_k]} \liminf_{n\to\infty} f_n'(x)dx = 0.$$

By Theorem 4.4 Monotone Convergence Theorem $(*4)$,

$$\begin{aligned} 0 = \lim_{k\to\infty} \int_{[a_k,b_k]} \liminf_{n\to\infty} f_n'(x)dx &= \lim_{k\to\infty} \int_{(0,1)} (\liminf_{n\to\infty} f_n'(x)) \cdot \chi_{[a_k,b_k]}(x)dx \\ &\overset{*4}{=} \int_{(0,1)} \lim_{k\to\infty} (\liminf_{n\to\infty} f_n'(x)) \cdot \chi_{[a_k,b_k]}(x)dx \\ &= \int_{(0,1)} \liminf_{n\to\infty} f_n'(x)dx. \end{aligned}$$

So we have

$$\int_{(0,1)} \liminf_{n\to\infty} f_n'(x)dx = 0.$$

Since $\liminf_{n\to\infty} f_n'(x)$ is non-negative, the above integral implies that $\liminf_{n\to\infty} f_n'(x) = 0$ a.e $x \in (0,1)$. (Review the properties derived from Definition 4.2.) Now the proof is complete.

□

9 (Exercise 3) Similar to Theorem 5.1 Vitali's Covering Theorem, we may suppose that every $I \in \Gamma$ is a closed inteval. Because if we obtain a countable disjoint closed intervals $\{I_j\}_{j=1}^{\infty}$ with $m(E \setminus \bigcup_{j=1}^{\infty} I_j) = 0$, then $\{\mathring{I}_j\}$ are countable disjoint open intervals and $m(E \setminus \bigcup_{j=1}^{\infty} \mathring{I}_j) = 0$. (Note that edge points are measure zero sets.)

STEP 1. By Vitali's Covering Theorem, we can find a finite number of closed intervals $\{I_{1,k}\}_{k=1}^{K_1}$ s.t

$$m^*\left(E \setminus \bigcup_{k=1}^{K_1} I_{1,k}\right) < 1.$$

STEP 2. Let $F_1 \stackrel{\text{def}}{=} \bigcup_{k=1}^{K_1} I_{1,k}$. Since $E \setminus F_1 \subset E$ and Γ is a Vitali cover of E, so Γ is also a Vitali cover of $E \setminus F_1$. Suppose that we have picked an arbitrary point $x \in E \setminus F_1$. We pick $I_{x,\delta} \in \Gamma$ with $x \in I_{x,\delta}$ and $\text{diam}(I_{x,\delta}) < \delta$. If we choose sufficiently small $\delta > 0$, then $I_{x,\delta} \cap F_1 = \emptyset$ because F_1 is closed. (Otherwise, we can find a sequence $\{x_n\} \subset I_{x,\delta} \cap F_1 \subset F_1$ with $x_n \to x$ by taking $\delta \to 0$. Then $x \in F_1$ because F_1 is closed. This contradicts to the assumption that $x \in E \setminus F_1$.)

Therefore, $\Gamma_1 \stackrel{\text{def}}{=} \{I \in \Gamma \mid I \cap F_1 = \emptyset\}$ is a Vitali cover of $E \setminus F_1$. By Vitali's Covering Theorem, we can find a finite number of closed intervals $\{I_{2,k}\}_{k=1}^{K_2} \subset \Gamma_1$ s.t

$$m^*\left((E \setminus F_1) \setminus \bigcup_{k=1}^{K_1} I_{2,k}\right) < \frac{1}{2}.$$

STEP 3. We continue the procedure in the similar way. Then we obtain disjoint closed intervals $\{I_{j,k}\}$ s.t

$$m^*\left(E \setminus \bigcup_{j=1}^{n} \bigcup_{k=1}^{K_j} I_{j,k}\right) < \frac{1}{n}.$$

for every $n \in \mathbb{N}$. Therefore

$$m^*\left(E \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{K_n} I_{n,k}\right) = 0.$$

Now the proof is complete.

□

10 (Exercise 4) Let

$$F(x) \stackrel{\text{def}}{=} f(x) - kx,$$

then $F(x)$ is also continuous on $[a, b]$. Note that

$$F(b) = F(a) \in \mathbb{R},$$

so $F(x)$ take the maximum value or the minimum value at some $x = x_0 \in (a, b)$. If $x = x_0$ is the maximizer of F, then

$$D^+F(x_0) \leqq 0 \leqq D_-F(x_0).$$

If $x = x_0$ is the minimizer of F, then

$$D^-F(x_0) \leqq 0 \leqq D_+F(x_0).$$

And then we have the desired conclusion. □

11 (Exercise 5)

STEP 1. We can find an open set $G_n \in \mathscr{O}^1$ with $E \subset G_n \subset [a, b]$ with $m(G_n) < \frac{1}{2^n}$. (Consider $\{I_{n,k}\}$ with $E \subset \bigcup_{k=1}^{\infty} I_{n,k}$ with $m^*(E) \leqq \sum_{k=1}^{\infty} |I_{n,k}| < m^*(E) + \epsilon_n$ where $\epsilon_n = \frac{1}{2^n}$. Let $G_n \stackrel{\text{def}}{=} \bigcup_{k=1}^{\infty} I_{n,k}$. Review Chapter 2.) Let

$$f_n(x) \stackrel{\text{def}}{=} m([a, x] \cap G_n).$$

Obviously, $0 \leqq f_n(x) \leqq m(G_n) < \frac{1}{2^n}$ and each $f_n(x)$ is a monotone increasing function on $[a, b]$. Furtheremore,

$$\begin{aligned} f_n(x+h) - f_n(x) &= m([a, x+h] \cap G_n) - m([a, x] \cap G_n) \\ &= m((x, x+h] \cap G_n) \leqq m((x, x+h]) = h, \end{aligned}$$

so each $f_n(x)$ is a continuous function.

STEP 2. Let us consider

$$S(x) = \sum_{n=1}^{\infty} f_n(x), \quad S_n(x) \stackrel{\text{def}}{=} \sum_{k=1}^{n} f_k(x).$$

Obviously $S(x)$ is non-negative and monotone increasing. Since $S_n(x)$ is continuous (because it is a sum of a finite number of continuous functions) and $S_n(x) \xrightarrow{u} S(x)$ converges uniformly on $x \in [a, b]$ (see below), $S(x)$ is continuous. (Recall that a sequence of continuous functions uniformly converges to a function, then the function is also continous.)

$$\begin{aligned} |S(x) - S_n(x)| &= S(x) - S_n(x) \\ &= \lim_{k \to \infty} (S_k(x) - S_n(x)) \\ &= \lim_{k \to \infty} \sum_{j=n+1}^{k} f_j(x) \\ &\leqq \lim_{k \to \infty} \sum_{j=n+1}^{k} \frac{1}{2^j} = \frac{1}{2^n} \end{aligned}$$

So

$$\lim_{n\to\infty}\sup_{x\in[a,b]}|S(x)-S_n(x)|=0.$$

In conclusion, $S(x)$ is a non-negative continuous and monotone increasing function.

STEP 3. We show that $S'(x)=\infty$ if $x\in E$. Since $E\subset G_n$, for every $x\in E$, $x\in G_n$. Since G_n is an open set, we can find $h_n>0$ s.t $[x,x+h_n]\subset G_n$. Let $h\overset{\text{def}}{=}\min\{h_1,\cdots,h_k\}$. Then $[x,x+h]\subset G_1,\cdots G_k$. Note that for $n=1,2,\cdots,k$,

$$\frac{f_n(x+h)-f_n(x)}{h}=\frac{m((x,x+h]\cap G_n)}{h}=\frac{m((x,x+h])}{h}=1,$$

so

$$\sum_{n=1}^{k}\frac{f_n(x+h)-f_n(x)}{h}=k.$$

Therefore

$$\begin{aligned}\frac{S(x+h)-S(x)}{h}&=\lim_{m\to\infty}\frac{S_m(x+h)-S_m(x)}{h}\\&=\lim_{m\to\infty}\sum_{n=1}^{m}\frac{f_n(x+h)-f_n(x)}{h}\\&\geqq\sum_{n=1}^{k}\frac{f_n(x+h)-f_n(x)}{h}=k.\end{aligned}$$

This implies that

$$\liminf_{h\to+0}\frac{S(x+h)-S(x)}{h}\geqq k.$$

Since k is an arbitrary natural number, by taking $k\to\infty$, we have

$$\lim_{h\to+0}\frac{S(x+h)-S(x)}{h}=\infty$$

By the similar argument above, we have

$$\lim_{h\to+0}\frac{S(x)-S(x-h)}{h}=\infty$$

(Consider $[x-h,x]\subset G_n$ for $n=1,\cdots,k$.) Now the proof is complete.

□

12 (Exercise 6) Let $\{r_n\}_{n\geqq1}\overset{\text{def}}{=}(0,1)\cap\mathbb{Q}$ and let

$$f_n(x)\overset{\text{def}}{=}\begin{cases}0 & x\in[0,r_n)\\ \frac{1}{2^n} & x\in[r_n,1]\end{cases}.$$

We claim that

$$S(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} f_n(x)$$

is the desired function. Note that $S(x)$ converges for all $x \in [0,1]$. This is because $0 \leqq S(x) \leqq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty$, and let $S_n(x) \stackrel{\text{def}}{=} \sum_{k=1}^{n} f_k(x)$, then $S_n(x) \leqq S_{n+1}(x)$ so $S(x) = \lim_{n\to\infty} S_n(x)$ exists.

STEP 1. First, we show that $S(x)$ is strictly monotone increasing. Since each $f_n(x)$ is monotone increasing, so $S(x)$ is monotone increasing. Let $x_1 < x_2 \in [0,1]$. There exists $r \in (x_1, x_2) \cap \mathbb{Q} \subset (0,1) \cap \mathbb{Q}$. This implies that there exists r_n s.t $x_1 \notin [r_n, 1]$ but $x_2 \in [r_n, 1]$, hence $f_n(x_1) = 0$ but $f_n(x_2) = \frac{1}{2^n}$. So $S(x_1) < S(x_2)$. It follows that $S(x)$ is strictly monotone increasing.

STEP 2. Next, we show that $S'(x) = 0$ a.e $x \in [a,b]$. Recall that each $f_n(x)$ is monotone increasing and $S(x)$ converges. So we can apply Theorem 5.3. Also note that $f_n'(x) = 0$ a.e $x \in [0,1]$, so $f_n'(x) = 0$ for all $n \in \mathbb{N}$ a.e $x \in [0,1]$. By Theorem 5.3,

$$S'(x) = \sum_{n=1}^{\infty} f_n'(x) = 0 \text{ a.e } x \in [0,1].$$

Now the proof is complete.

□

13 (Exercise 7) □

CHAPTER 6

Solutions